Cycle Therapy

A Prescription for Fold and Unfold on Regular Trees

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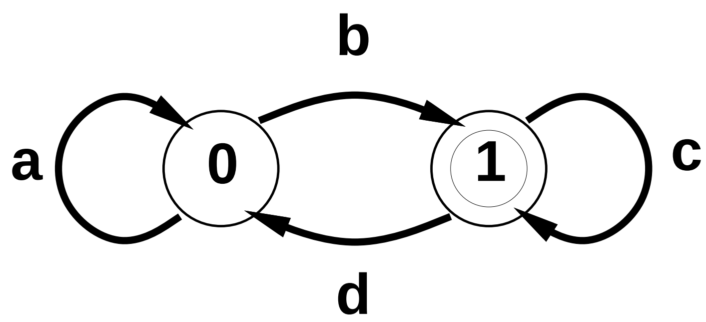
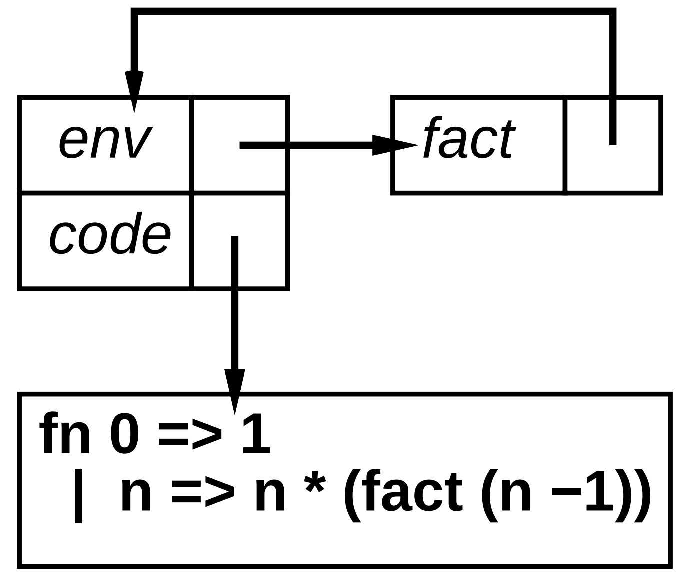
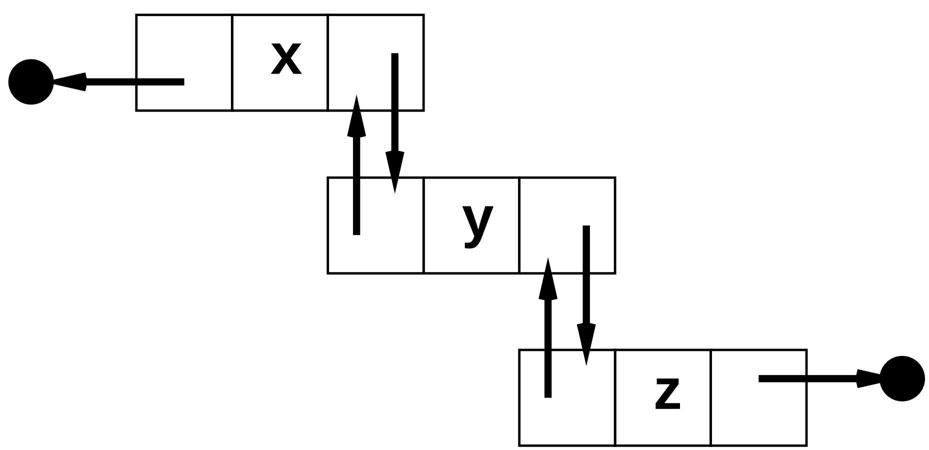
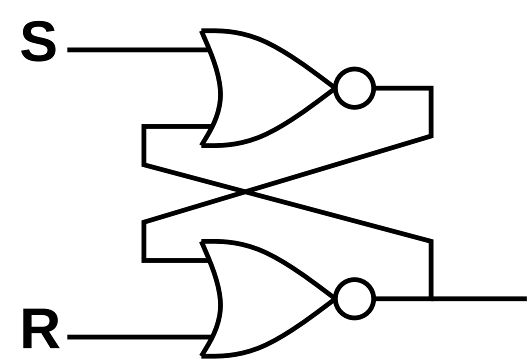
Cyclic Structures Are Ubiquitous









Digression: Strictness

Let $\bot$ (pronounced "bottom") stand for a computation which diverges (e.g., loops infinitely) or signals an error.

A mathematical function is *strict* in a parameter if the function returns $\bot$ whenever that parameter is $\bot$.

Examples:

- The + operator is strict in both arguments.
- The function `f(x,y) = x` is strict in the x parameter but non-strict in the y parameter.

Digression: Eagerness vs. Laziness

- An *eager* language models all programming language functions as mathematical functions that are strict in all parameter positions. E.g., the previous `f` would be treated as if it were written:

  ```
  f(x,y) = (if y == ⊥ then ⊥ else x)
  ```

  Most programming languages are eager. E.g.: Java, C, C++, Pascal, Fortran, Scheme, ML, ...

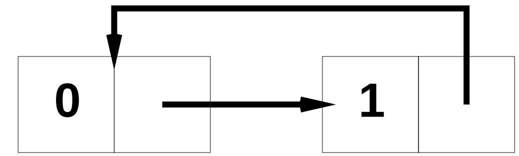
- A *lazy* language models programming language functions with their “natural” strictness. In particular, all data constructors are non-strict in all arguments. E.g.:

  ```
  f(3,(loop)) = 3
  length((loop):(loop):[]) = 2
  ```

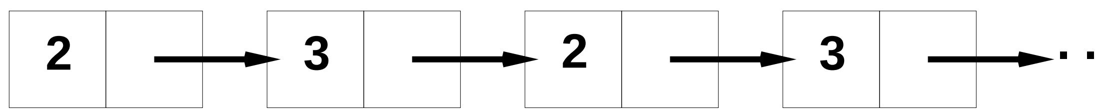
  Haskell is an example of a lazy language.

Cycles Are Tricky To Manipulate

Consider Haskell's `alts = 0:1:alts`



- Naïve generation ⇒ unbounded structures:
  - **let** `inf x y = x:(inf y x)` **in** `inf 2 3`



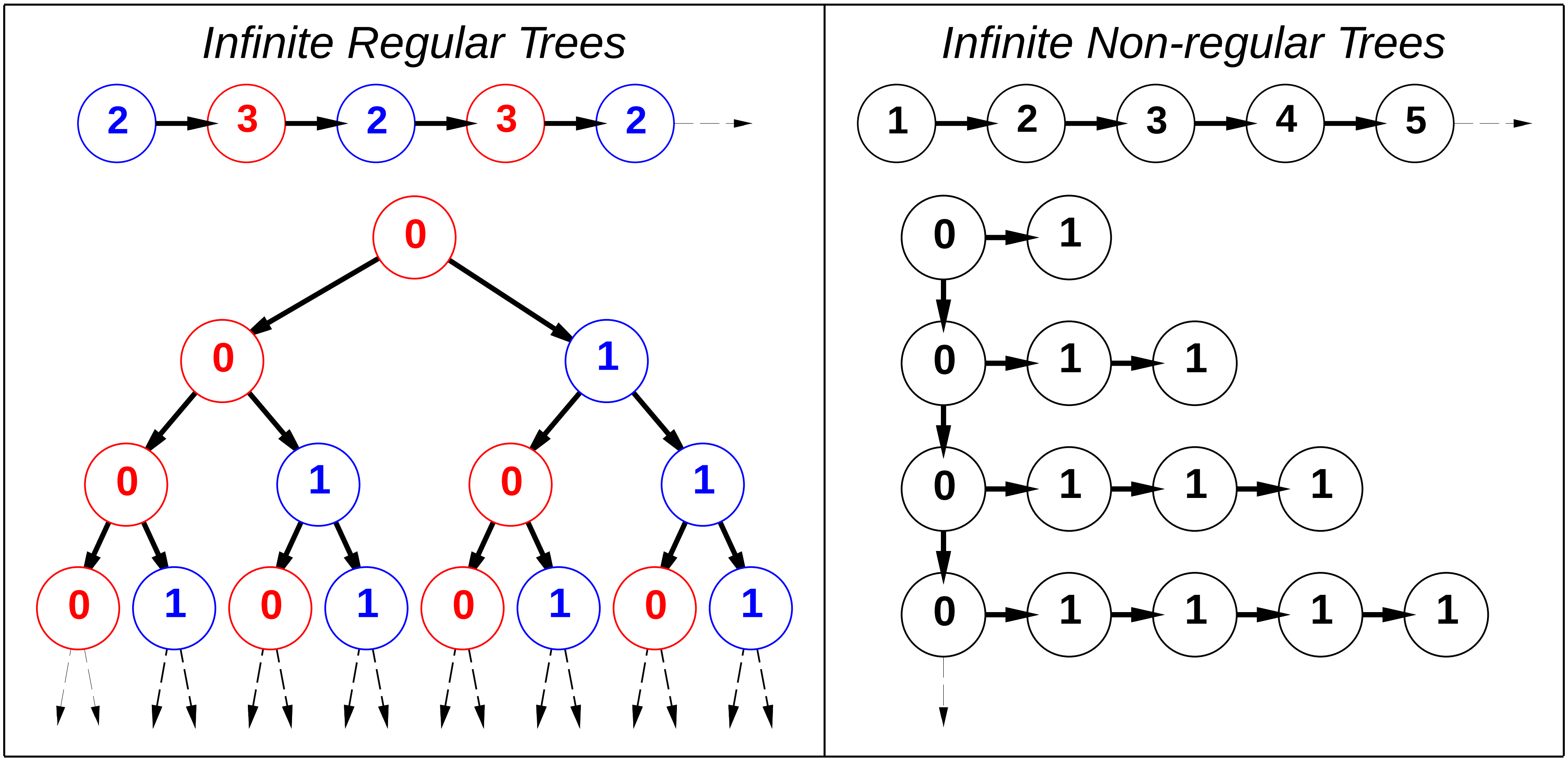
  - `map (\ x -> x + 2) alts`
- Naïve accumulation ⇒ divergence:
  - `foldr (+) 0 alts`
  - `foldr Set.insert Set.empty alts`
- Dependency on language features: laziness, side effects, node equality, recursive binding constructs, etc.

Road Map

- Viewing cyclic structures as infinite regular trees.
- Adapting the tree-generating unfold function to generate cyclic structures for infinite regular trees.
- Adapting the tree-accumulating fold function to return non-trivial results for strict combining functions and infinite regular trees.
- Cycamores: an abstraction for manipulating regular trees that we have implemented in ML and Haskell.

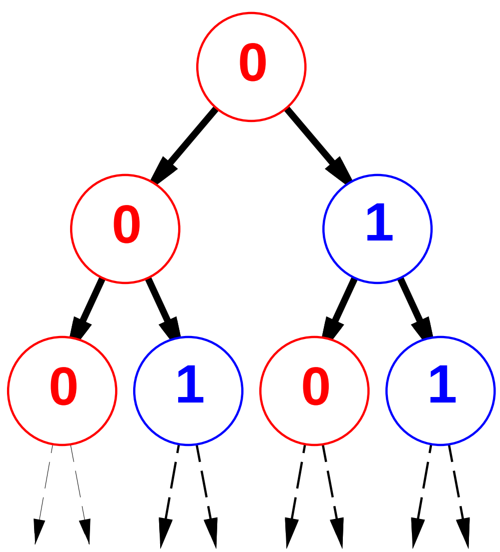
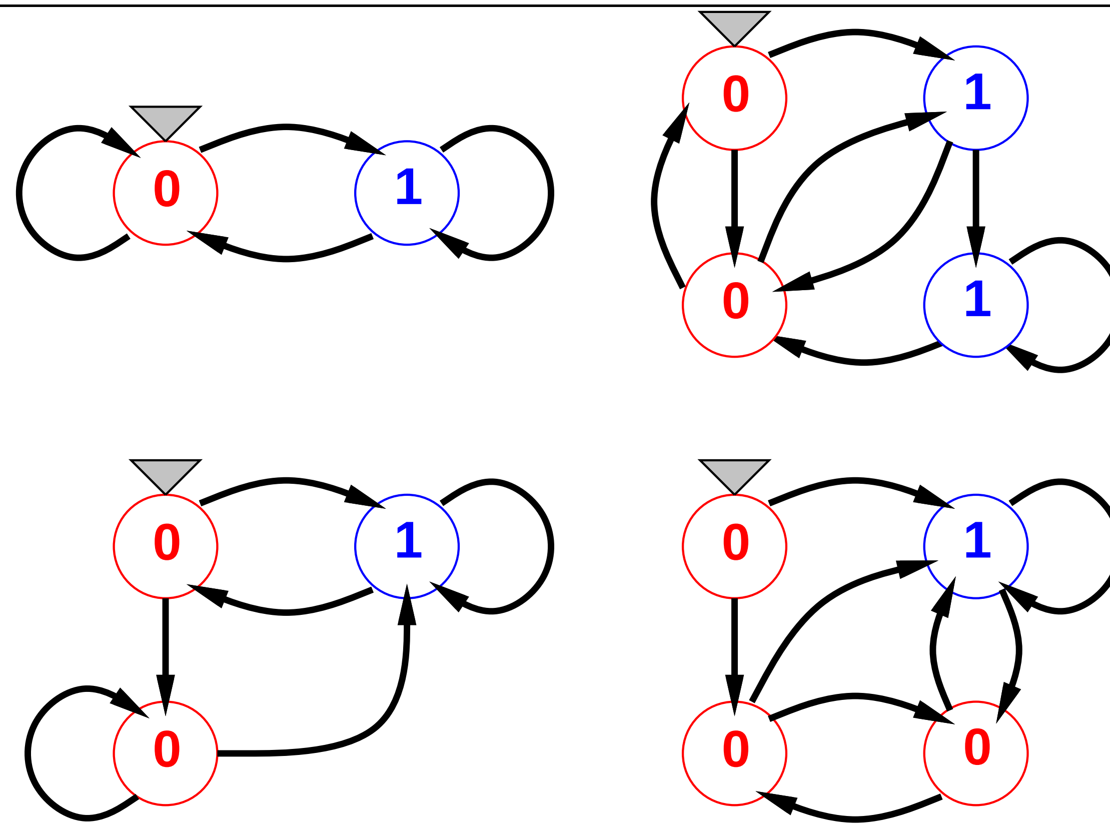
Regular Trees

A tree is *regular* if it has a finite number of distinct subtrees.



Cyclic Representatives

Finite cyclic graphs denote infinite regular trees.
The same tree may be represented by many finite graphs.

Infinite Regular Tree	*Some Finite Cyclic Representatives*
	

Goals

Develop high-level abstractions for creating and manipulating regular trees that:

- efficiently represent regular trees using cyclic graphs;
- do not expose the finite representative denoting an infinite regular tree;
- are relatively insensitive to the features of the programming language in which they are embedded.

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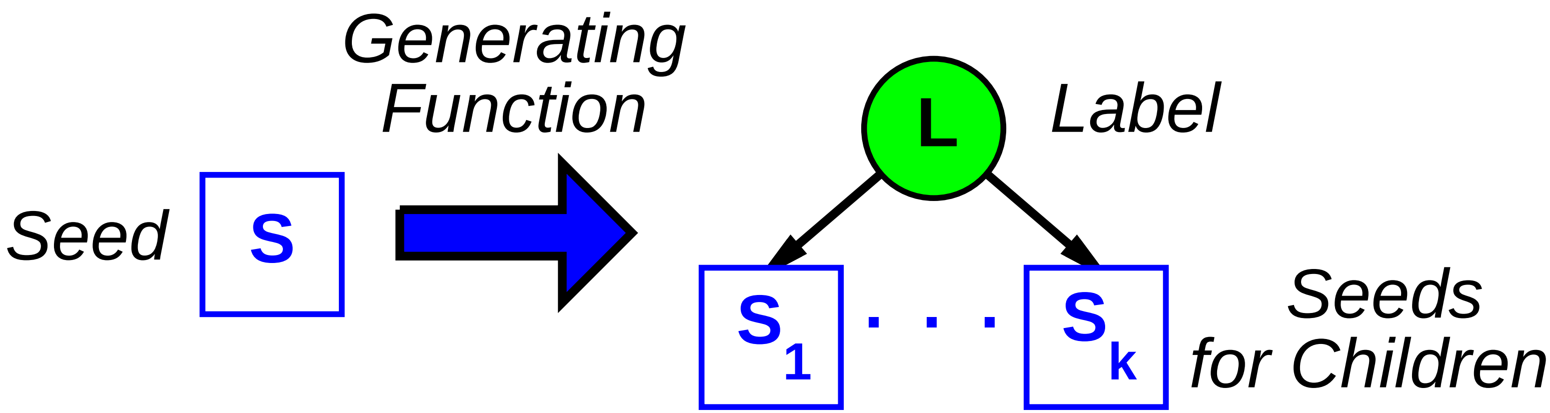
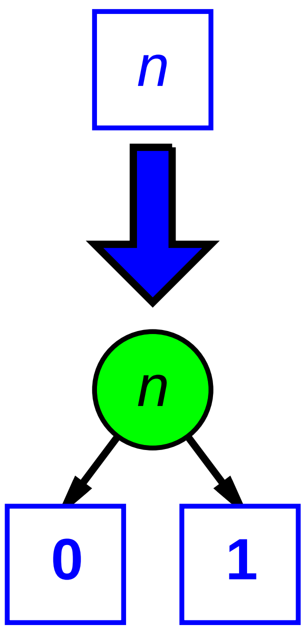
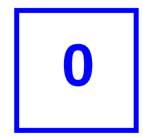
Tree Generation via Unfold

The `unfold` operator generates a tree from a generating function and a seed.



$$\text{unfold} : \underbrace{(S \rightarrow (L \times (S^{\omega})))}_{\text{generating function } \psi} \rightarrow \underbrace{\underbrace{S}_{\text{seed}} \rightarrow \underbrace{\text{Tree}(L)}_{\text{trees over L}}}_{\psi\text{-anamorphism}}$$

Unfold Example 1: Regular Tree

Generating Function ψ	*Tree*
	

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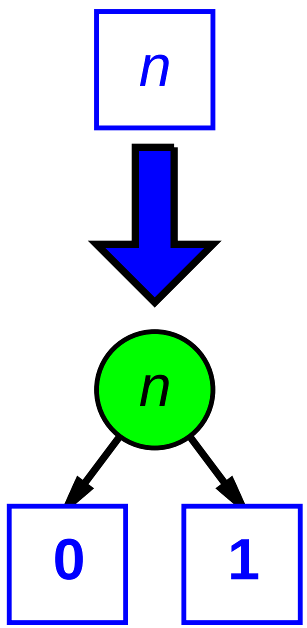
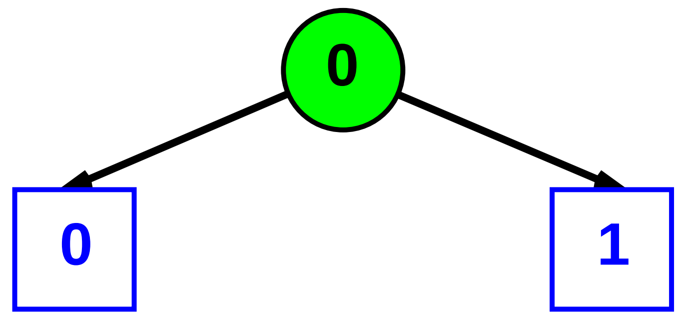
<tr><td>
</td><td>
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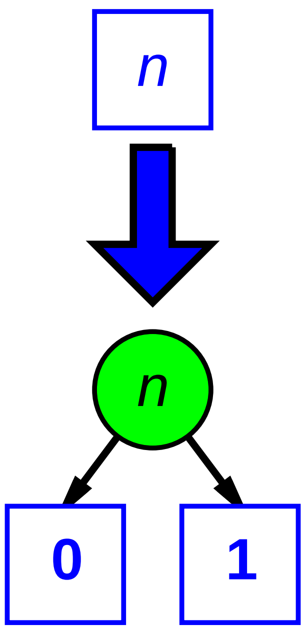	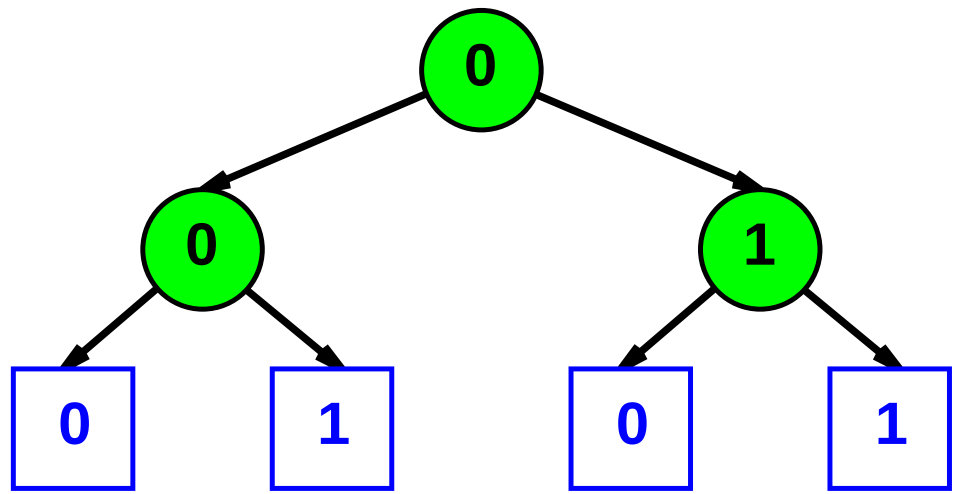

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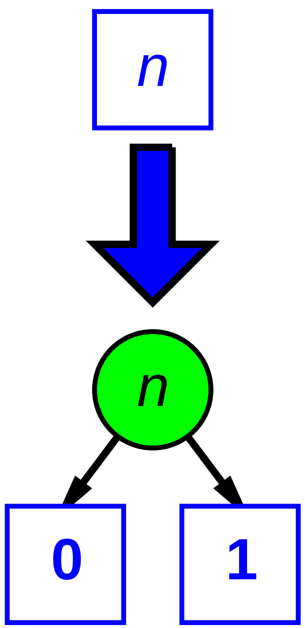	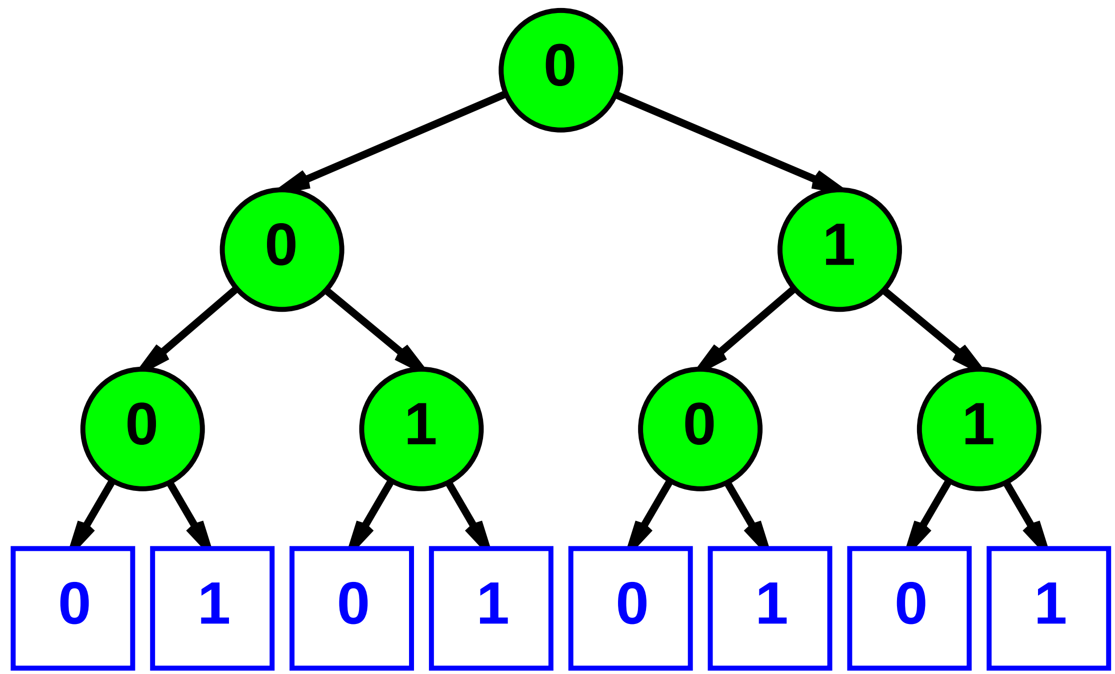

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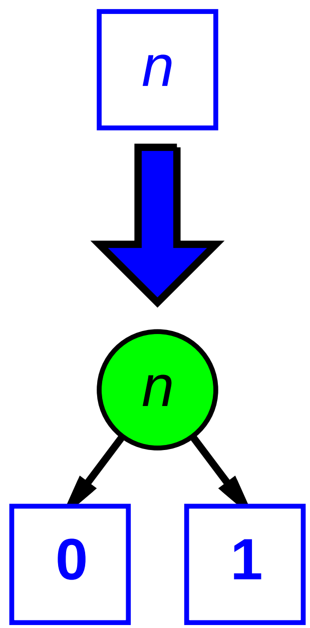
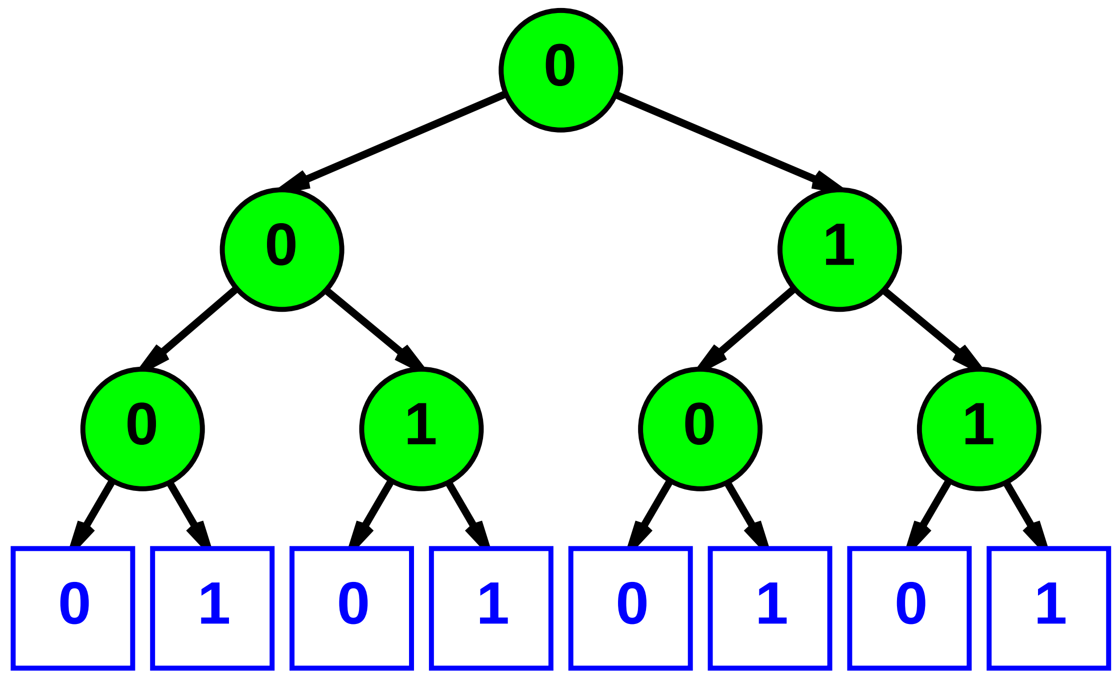
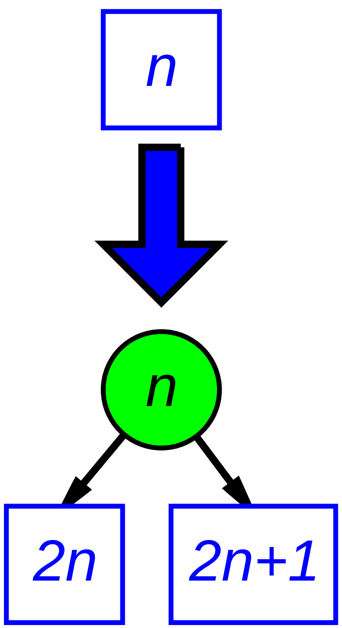
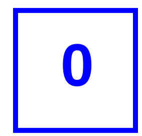
	

$$\mathsf{deps}(0, \psi) = \{0, 1\}$$

Unfold Example 2: Non-regular Tree

Generating Function ψ	*Tree*
	

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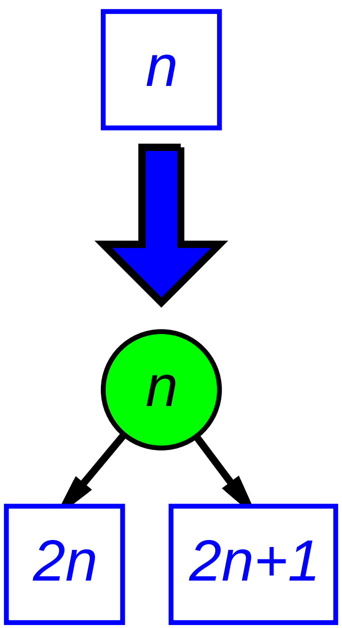
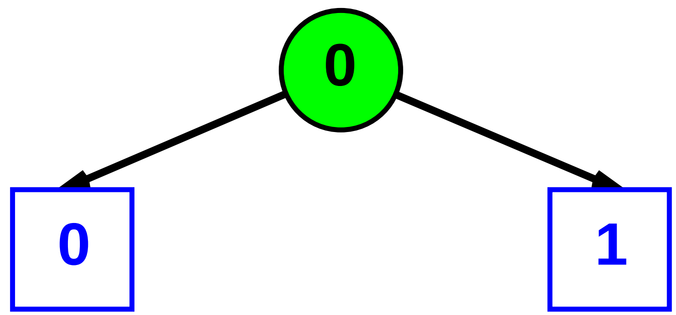
<tr><td>
</td><td>
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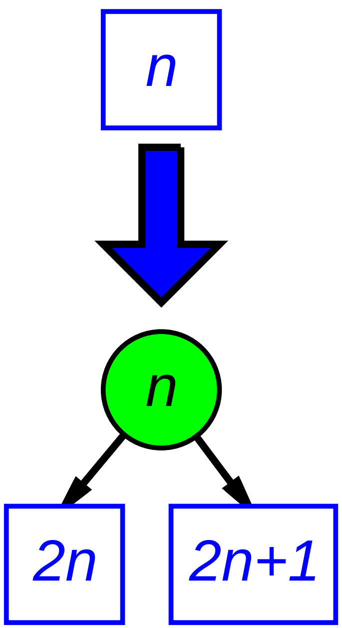	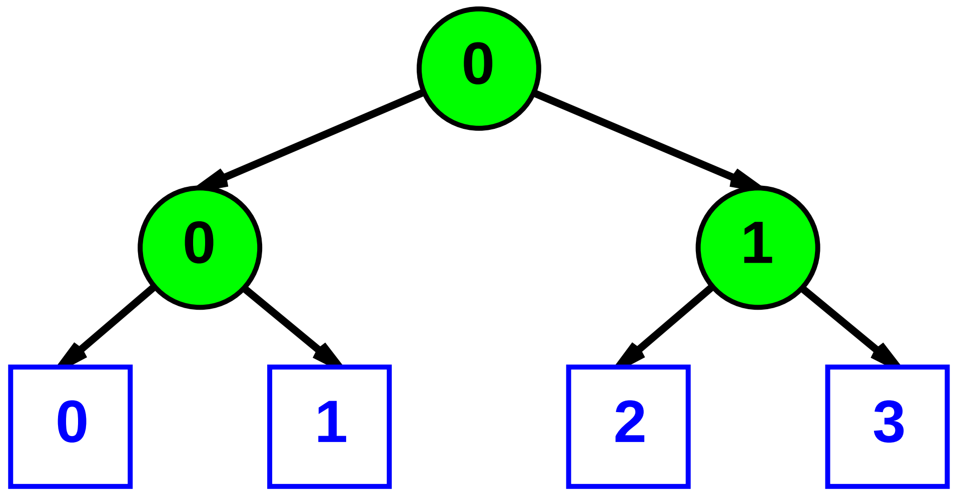

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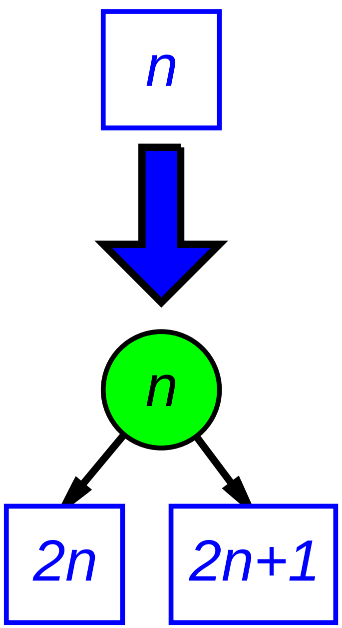	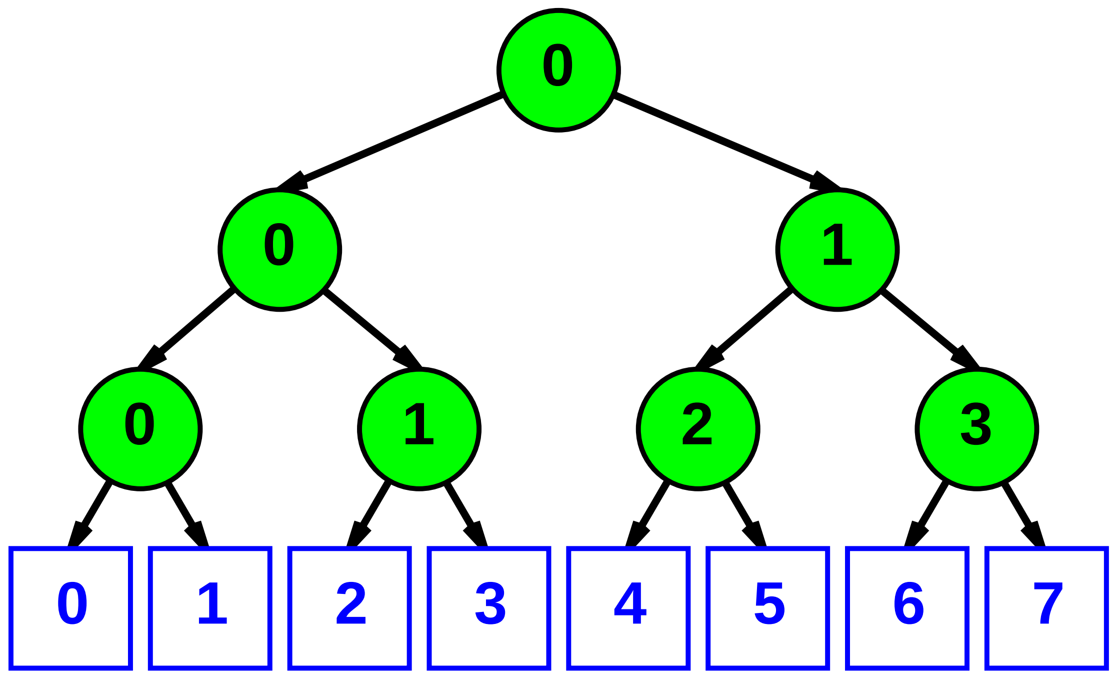

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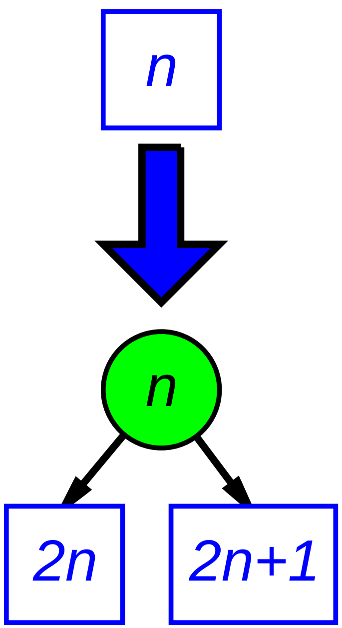	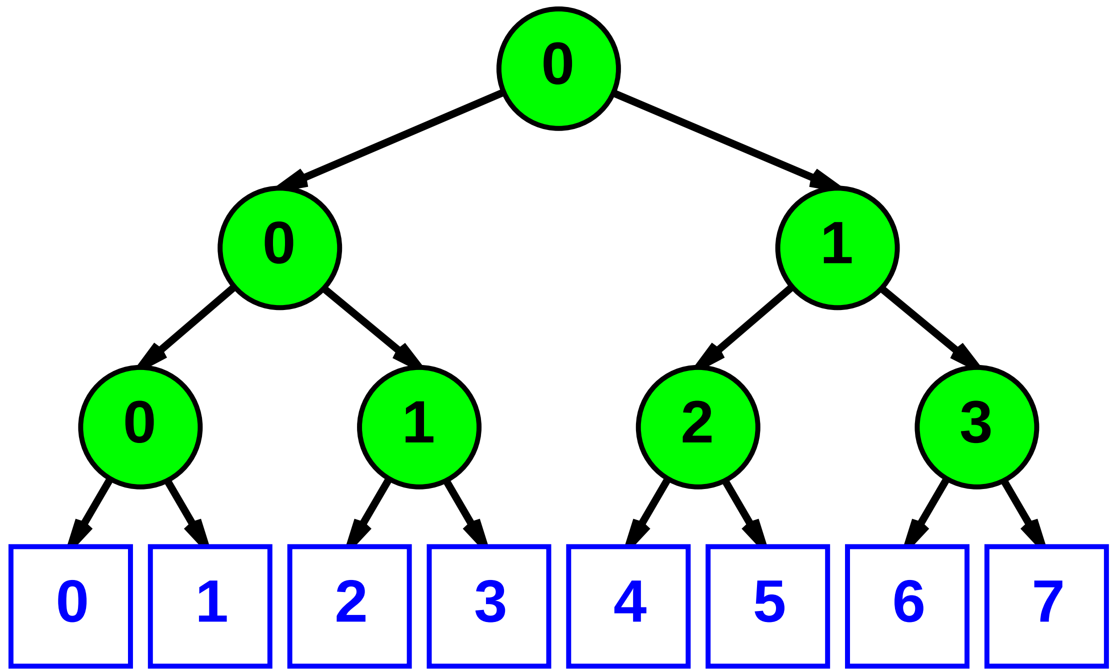

$$\mathsf{deps}(0, \psi) = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$$

Unfold Lemma

If $\mathsf{deps}(x, \psi)$ is finite, then $\mathsf{unfold}(\psi)(x)$ is a regular tree.

- Converse of this lemma does not hold.
- Basis for implementation of unfold that “ties cyclic knots” for (some) regular trees via memoization on seeds (a la Hughes’s *Lazy Memo Functions*, FPCA’85).

Unfold Implementation: Standard ML

generating fcn.: **fun** `P n = (n,[(n+1) mod 2])`

initial seed : `0`

unfold P 0

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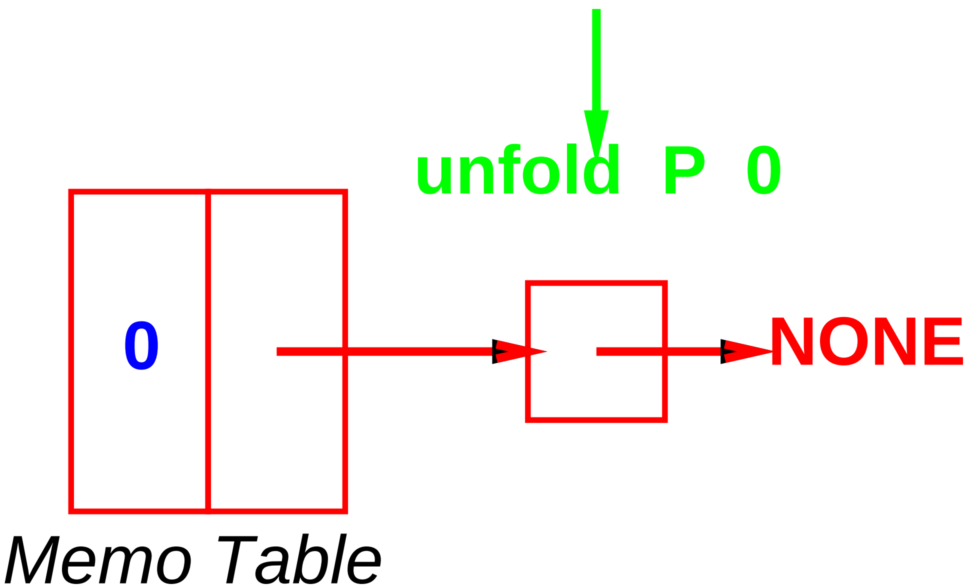

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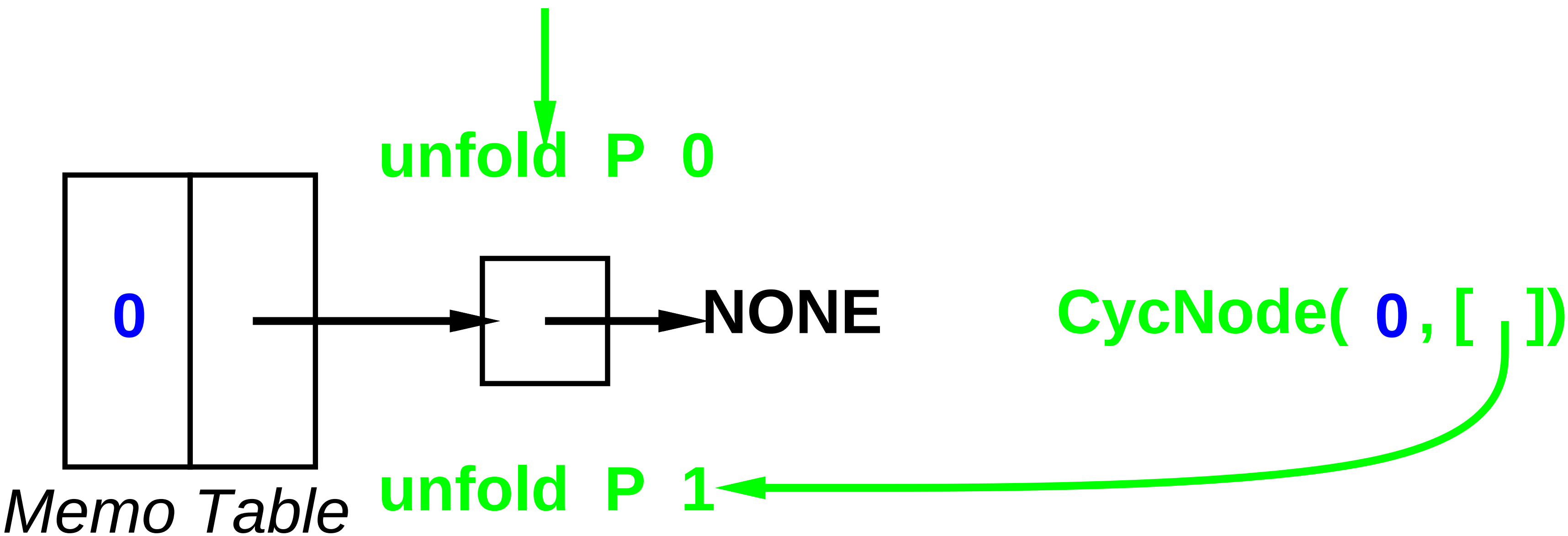

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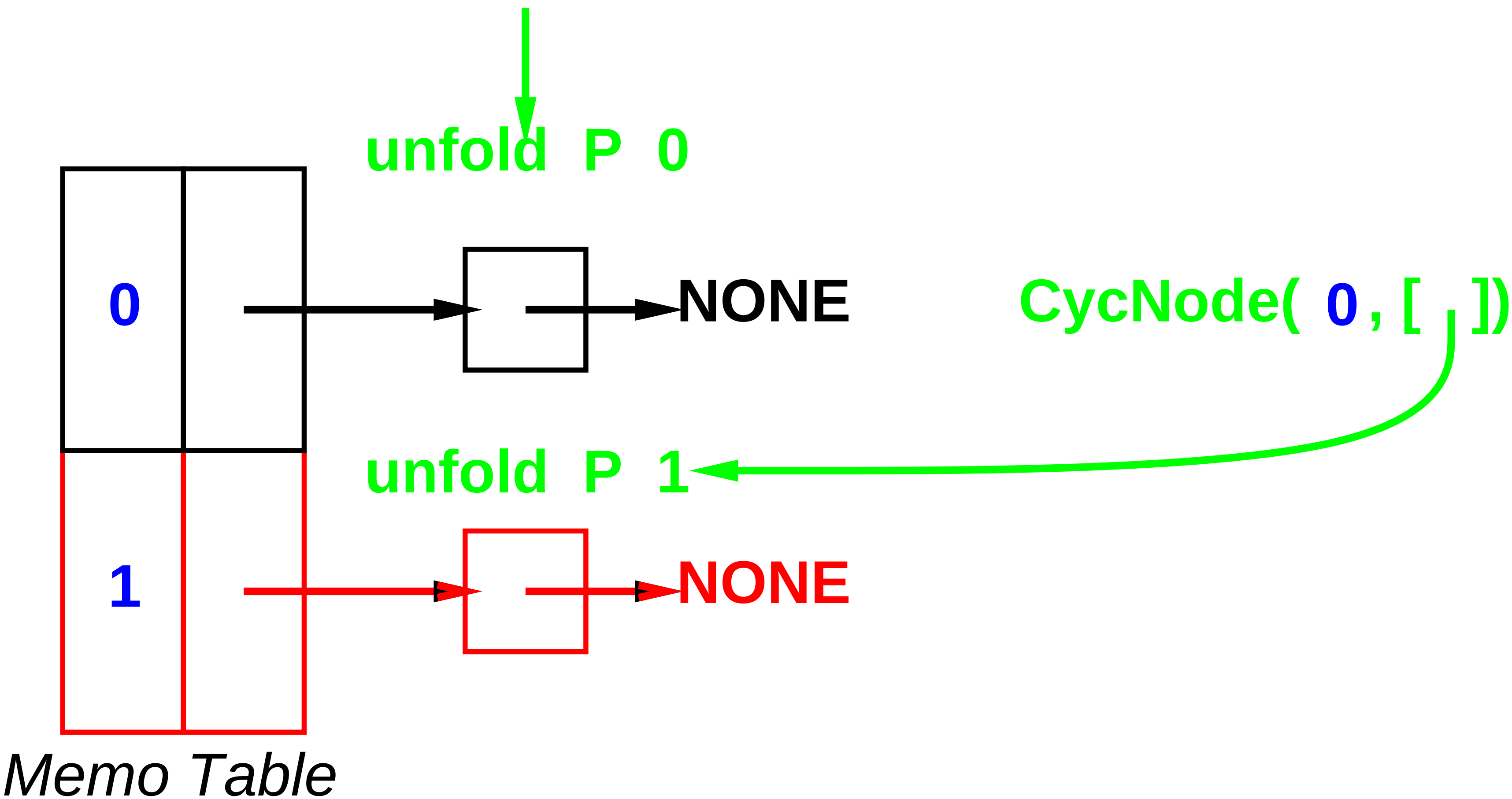

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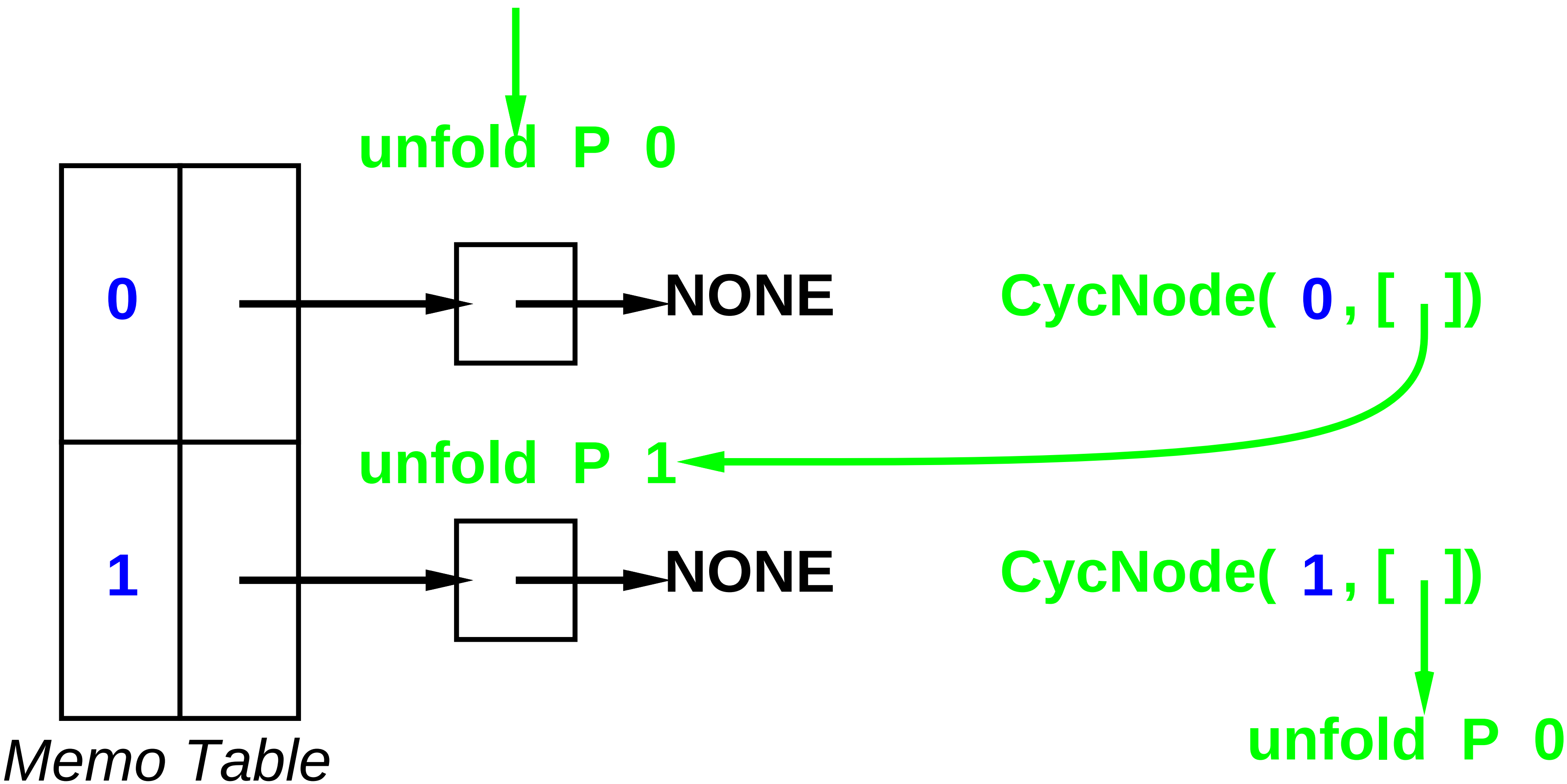

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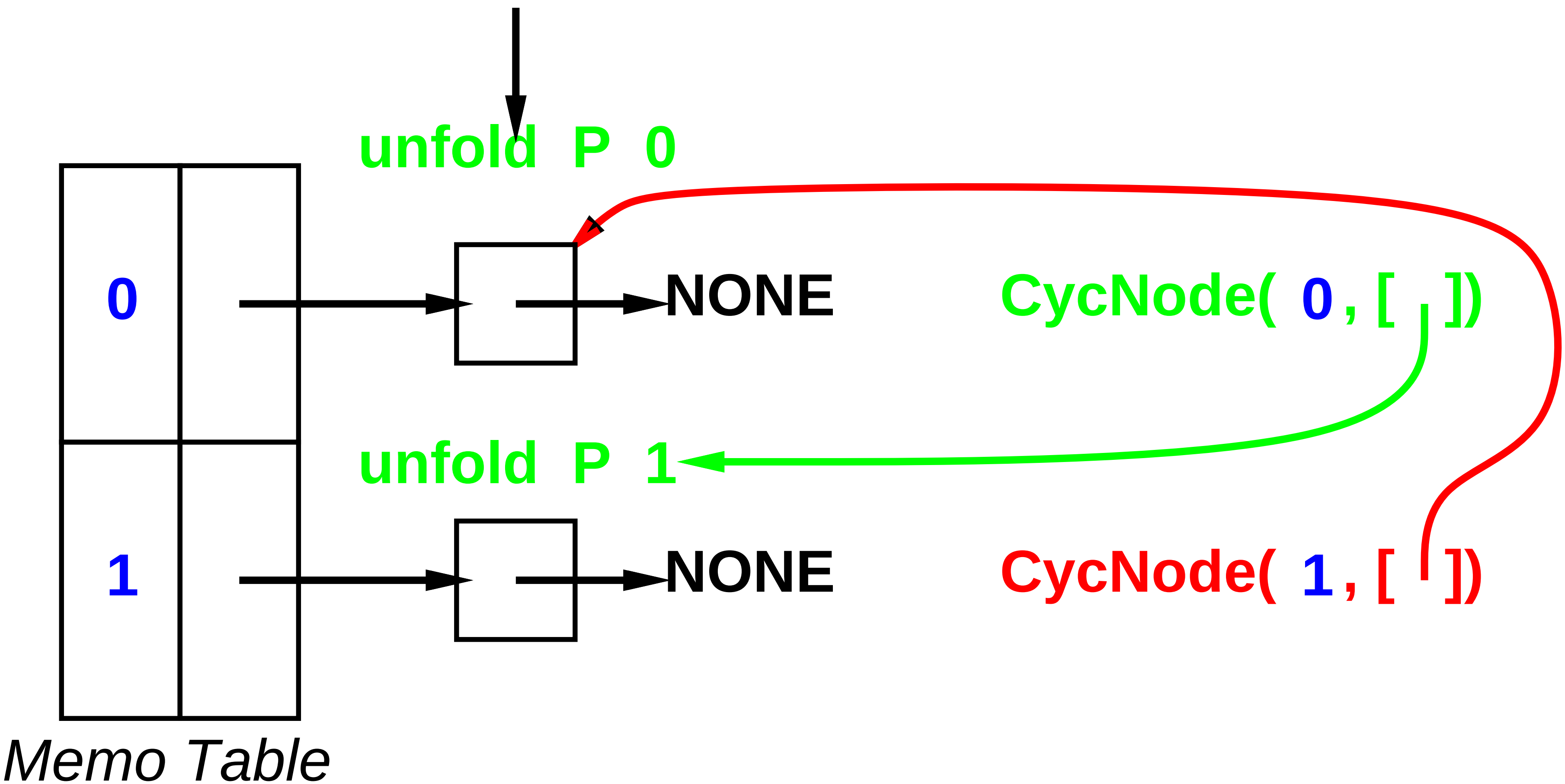

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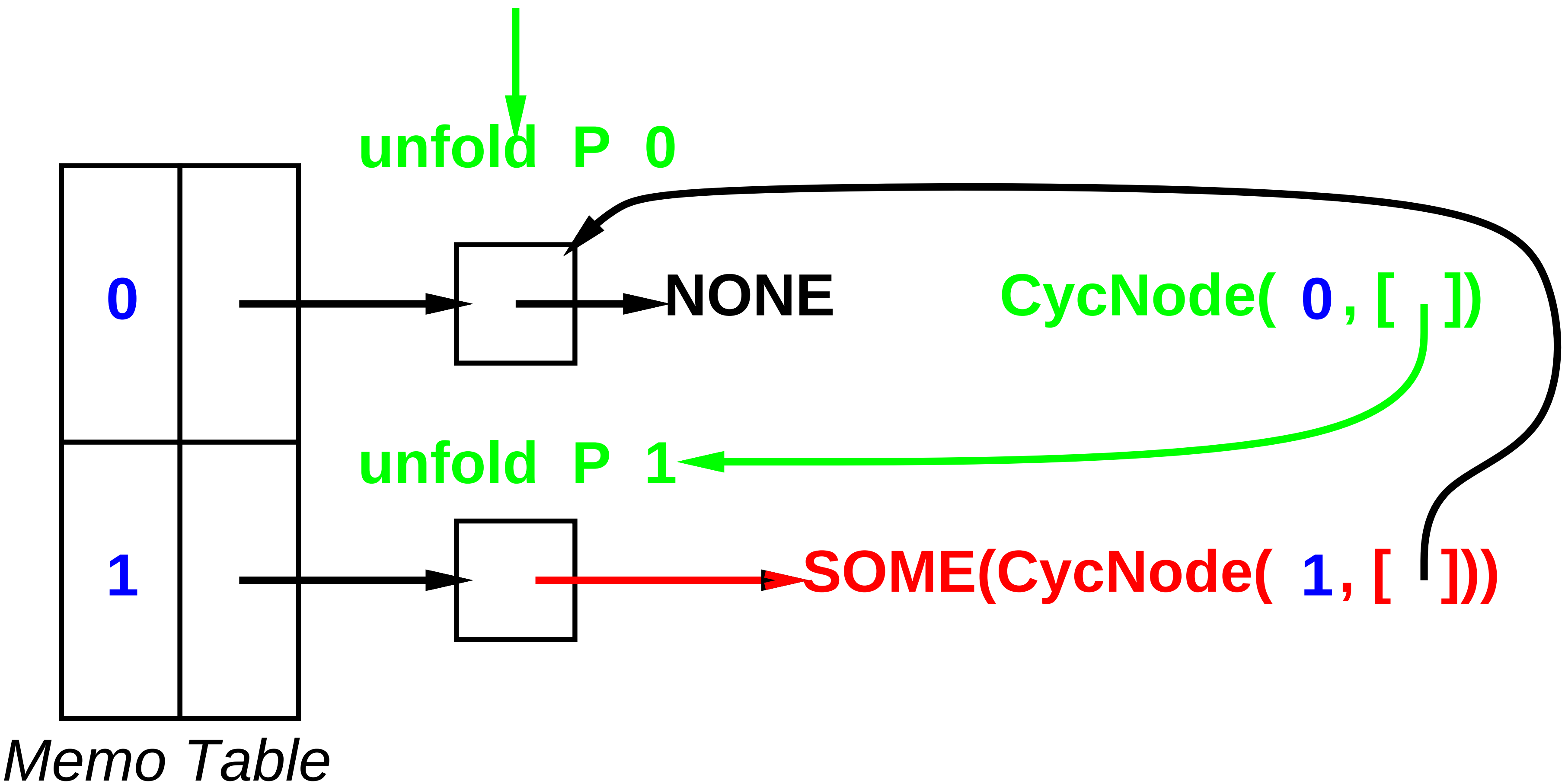

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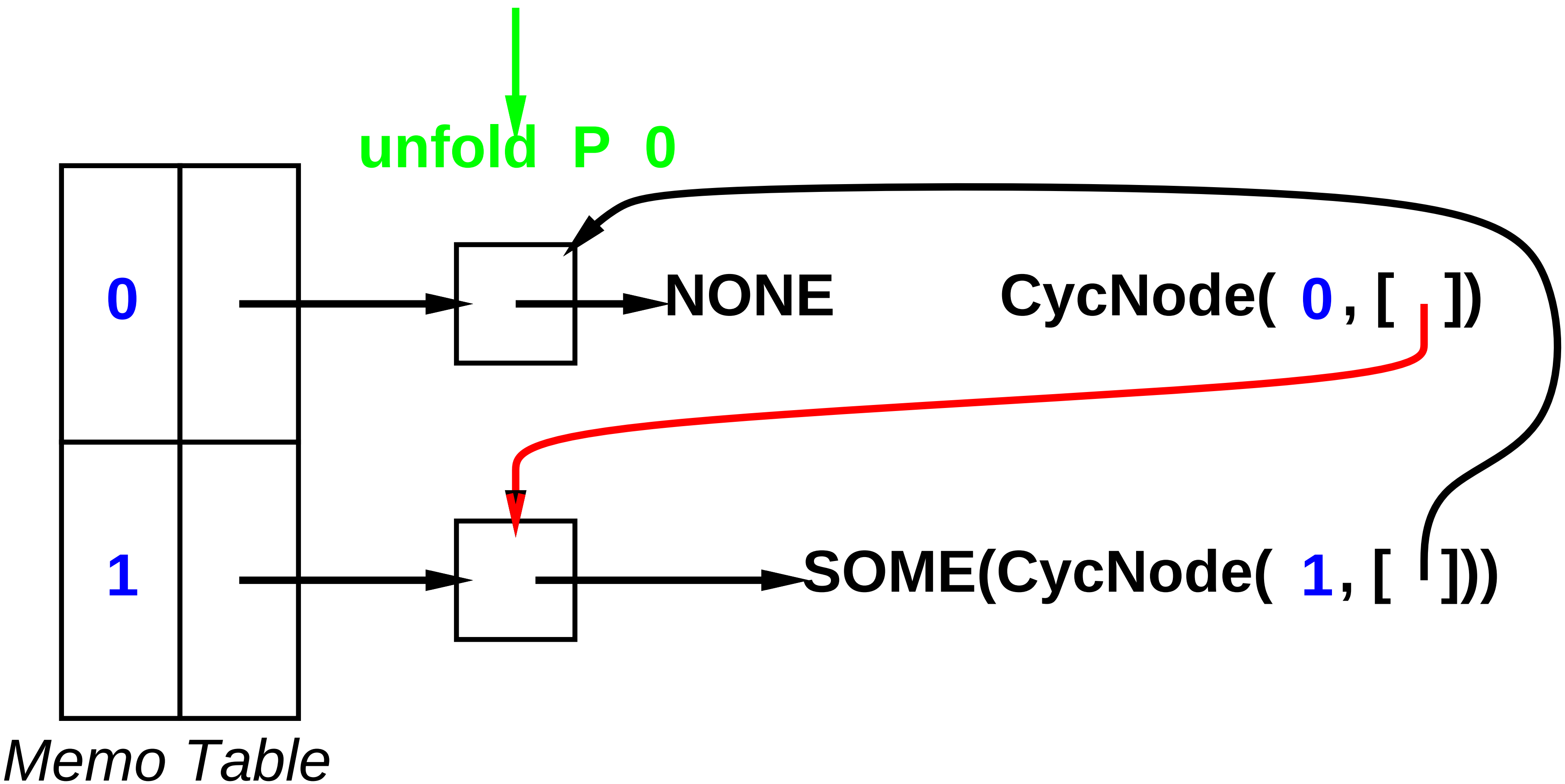

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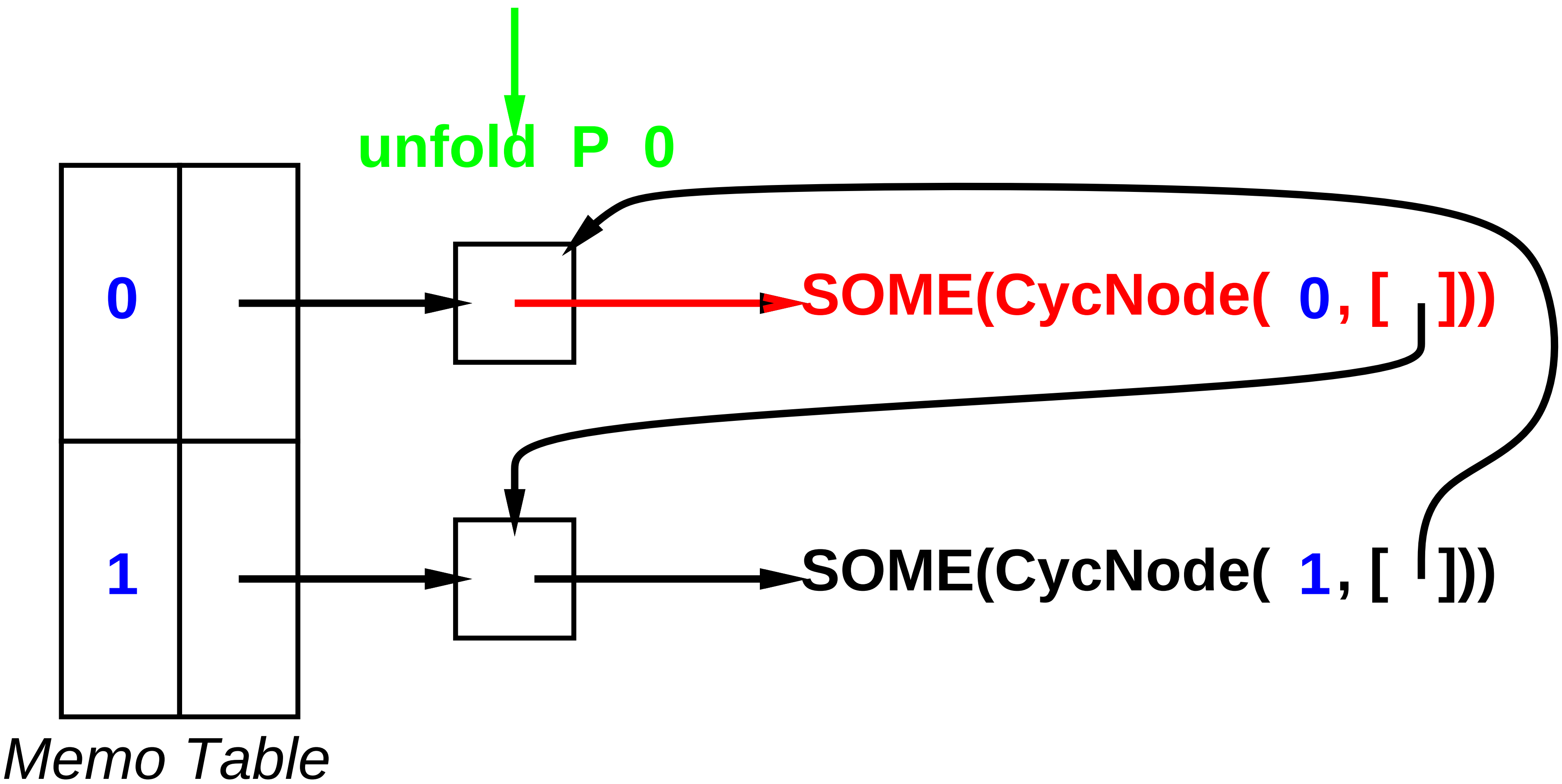

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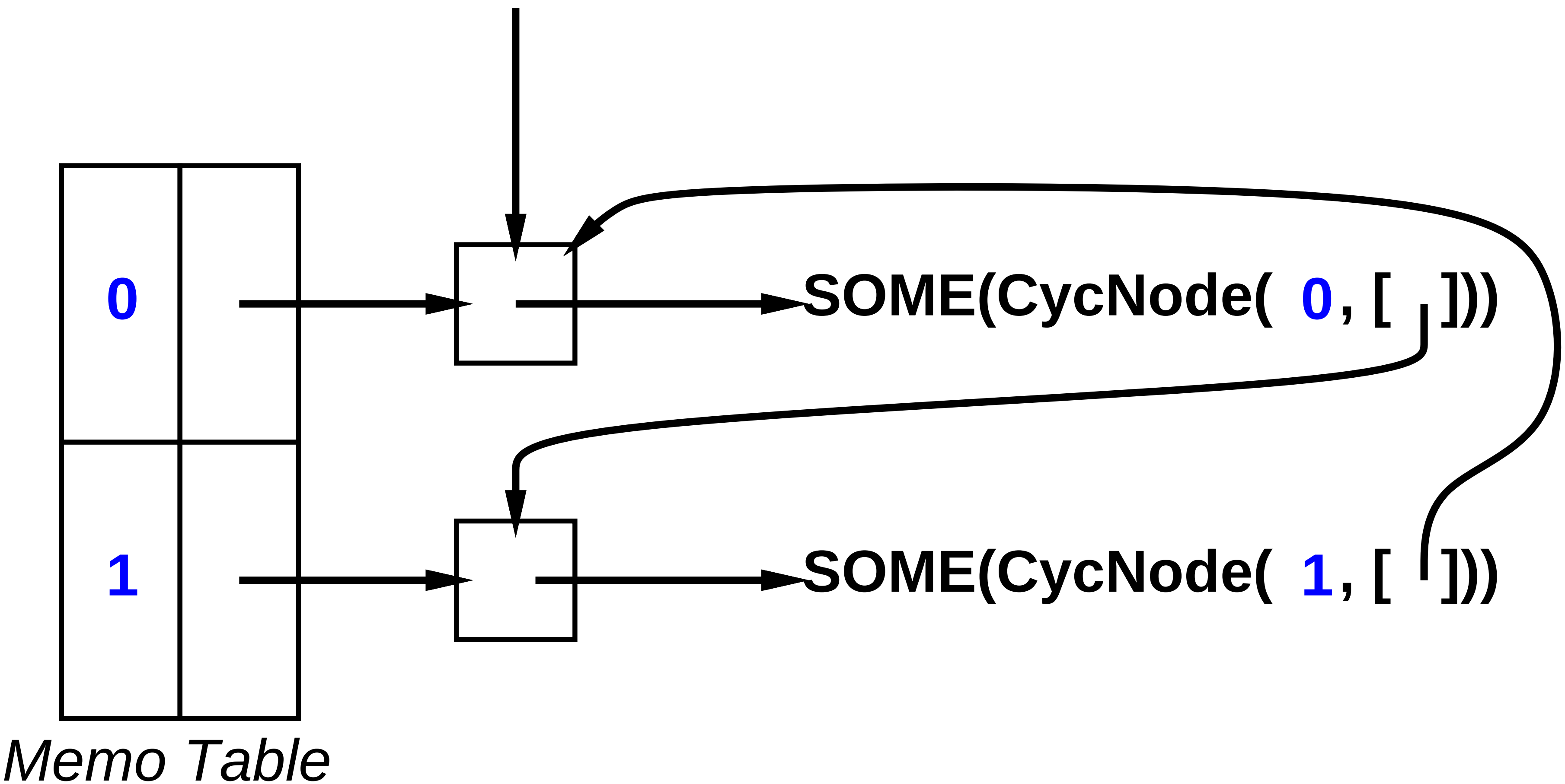

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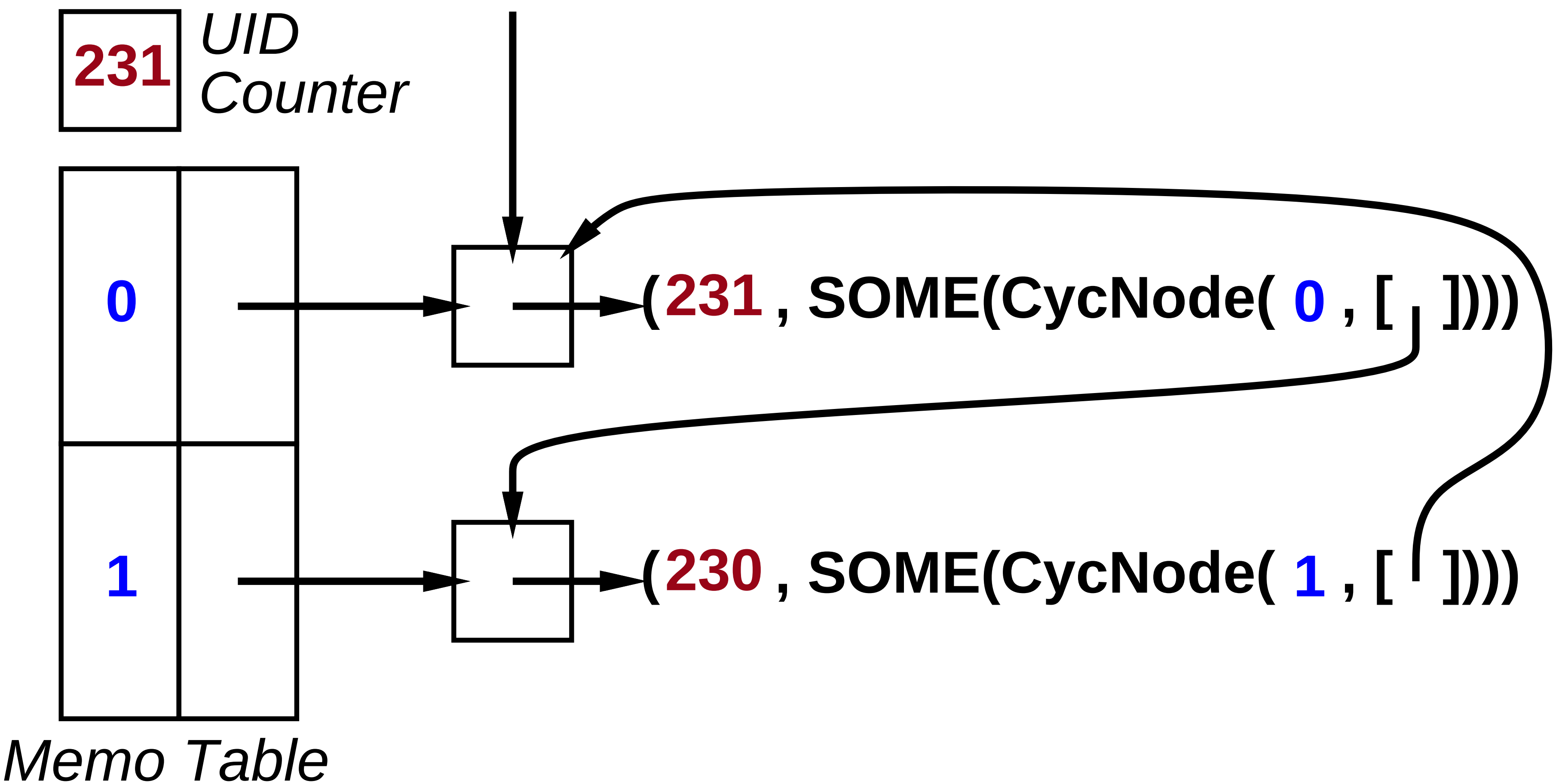

Unfold Implementation: Discussion

- Can use fewer reference cells in SML implementation.
- Cyclic hash-consing yields minimal graphs (Mauborgne, ESOP 2000; Considine & Wells, unpublished).
- Haskell implementation:
  - Uses laziness to tie cyclic knots.
  - Uses a `Cycle` monad to thread UID counter and memoization tables through computation.
  - Tricky to tie cyclic knots in presence of monad; use techniques of Erkok and Launchbury (ICFP '00).
- In practice, a memofix function is more flexible than unfold (see paper).

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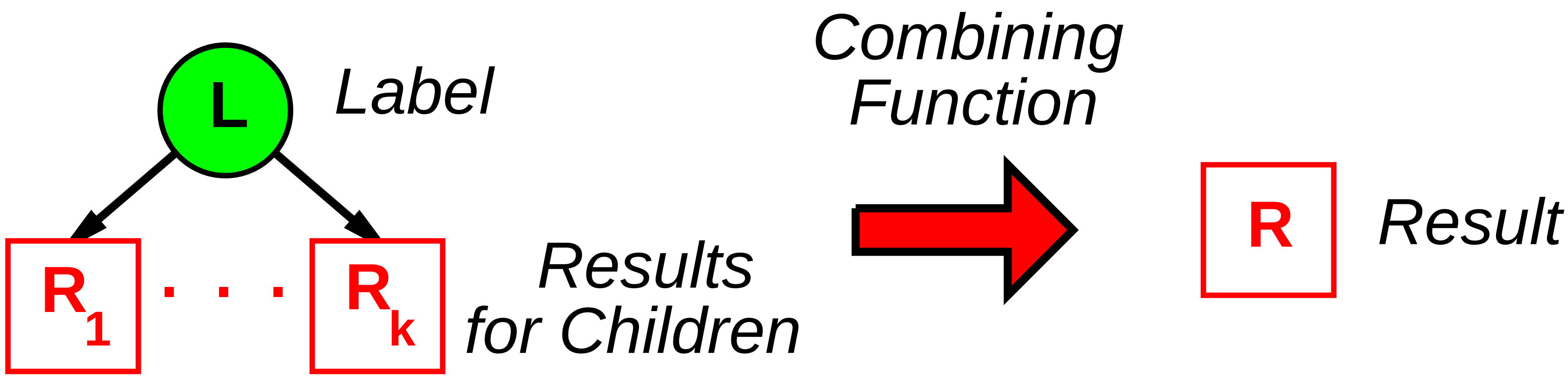
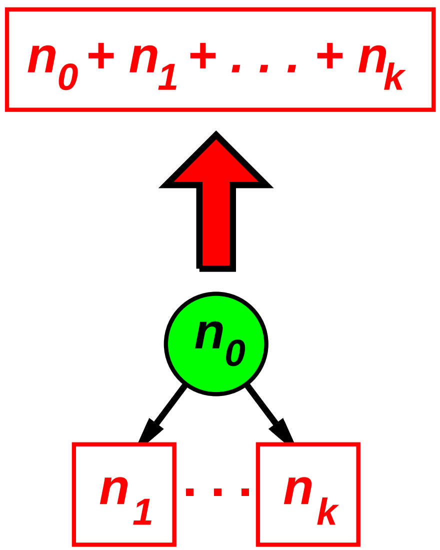
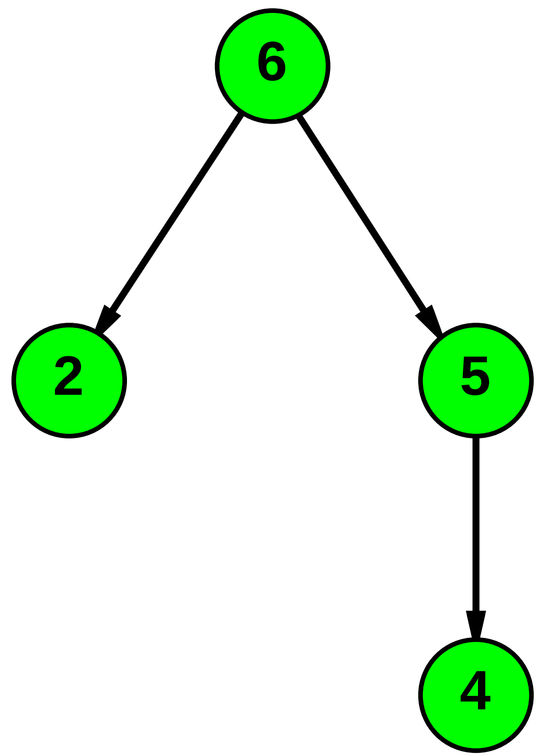
Tree Accumulation via Fold

The fold operator accumulates a result from a tree using a combining function.



$$\text{fold} : \underbrace{((L \times (\mathcal{C}_{\text{res}}{}^{\omega})) \xrightarrow{\text{cont}} \mathcal{C}_{\text{res}})}_{\text{accumulating function } \phi} \longrightarrow \underbrace{\text{Tree}(L) \xrightarrow{\text{cont}} \mathcal{C}_{\text{res}}}_{\substack{\text{tree valuation } \theta \\ (\phi\text{-catamorphism})}}$$

Folding over a Finite Tree

Accumulating Function ϕ	*Tree*
	

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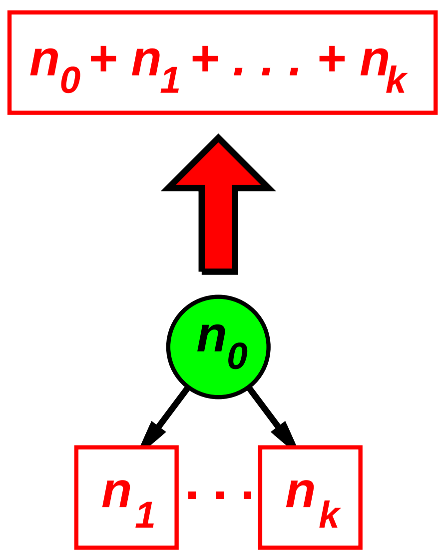	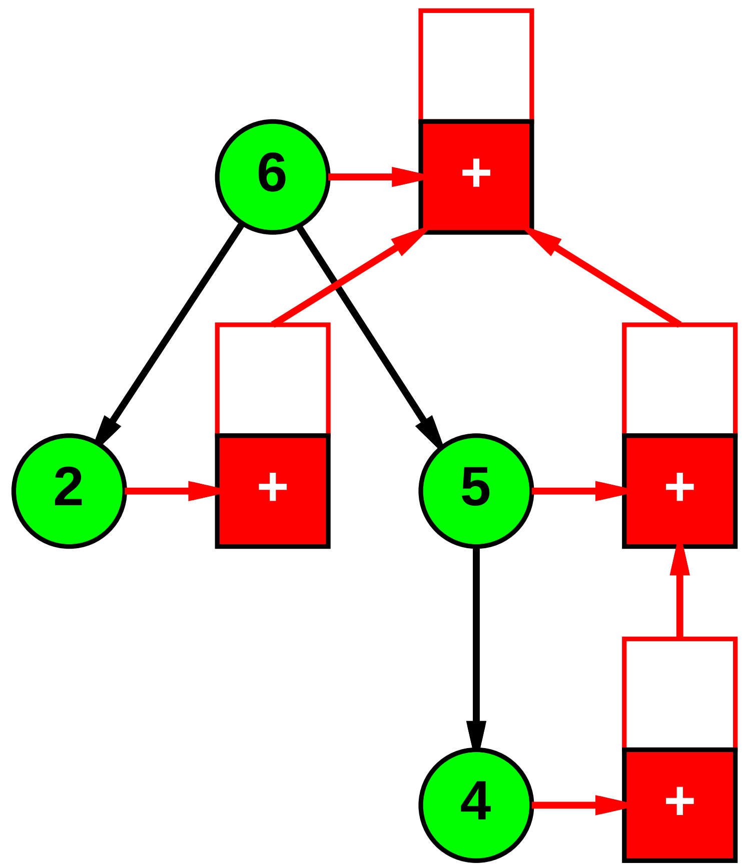

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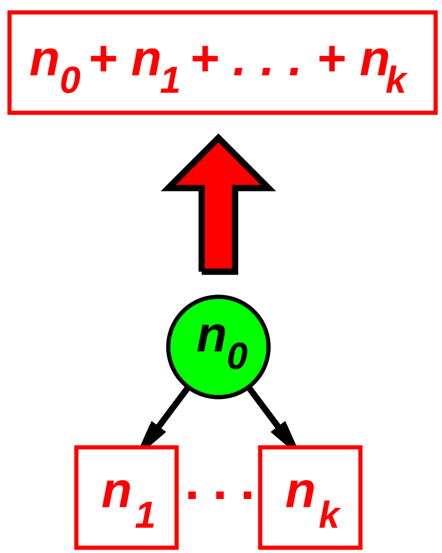	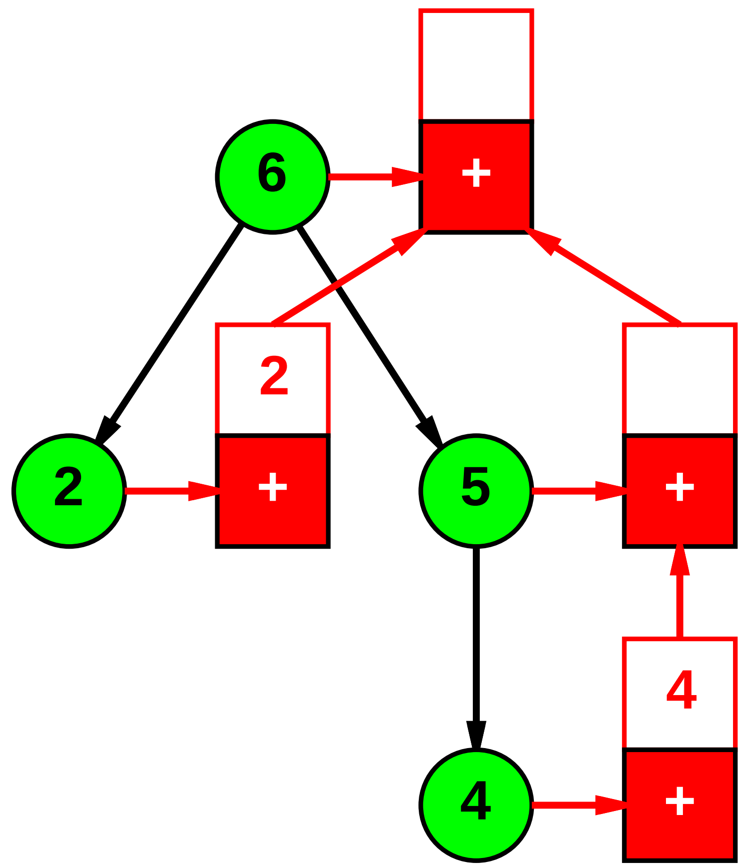

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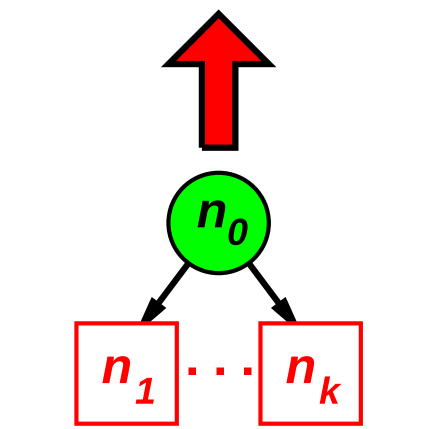	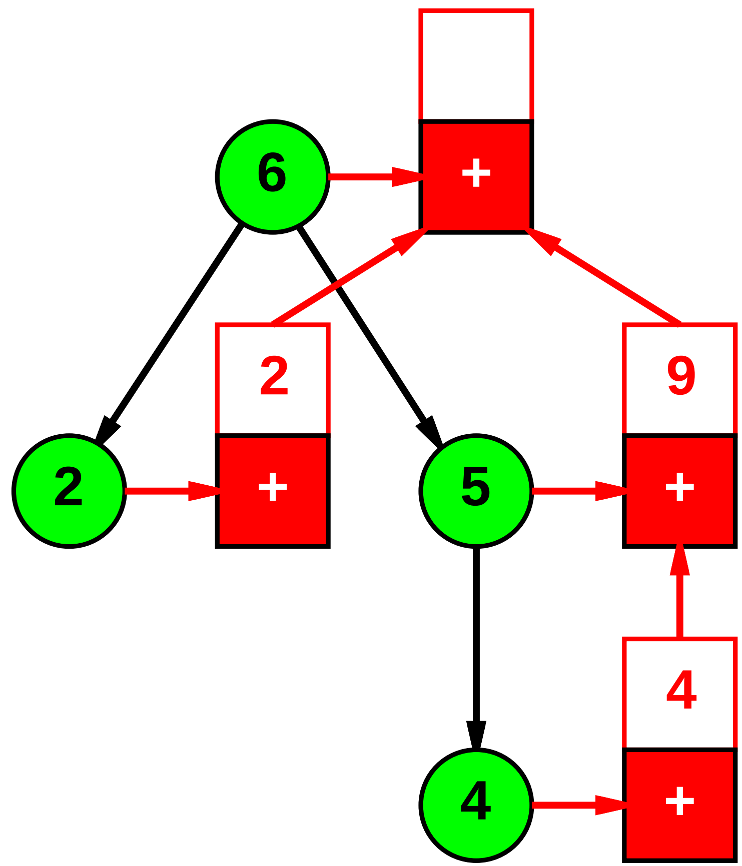

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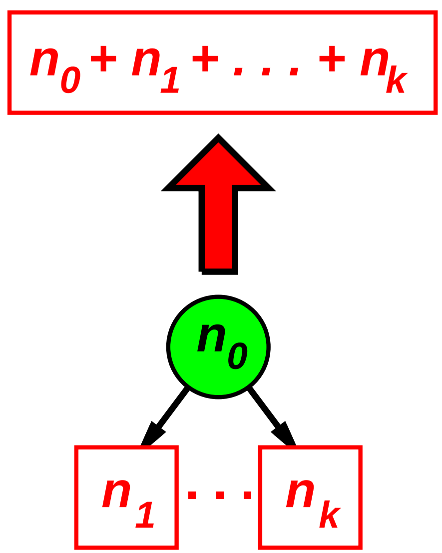
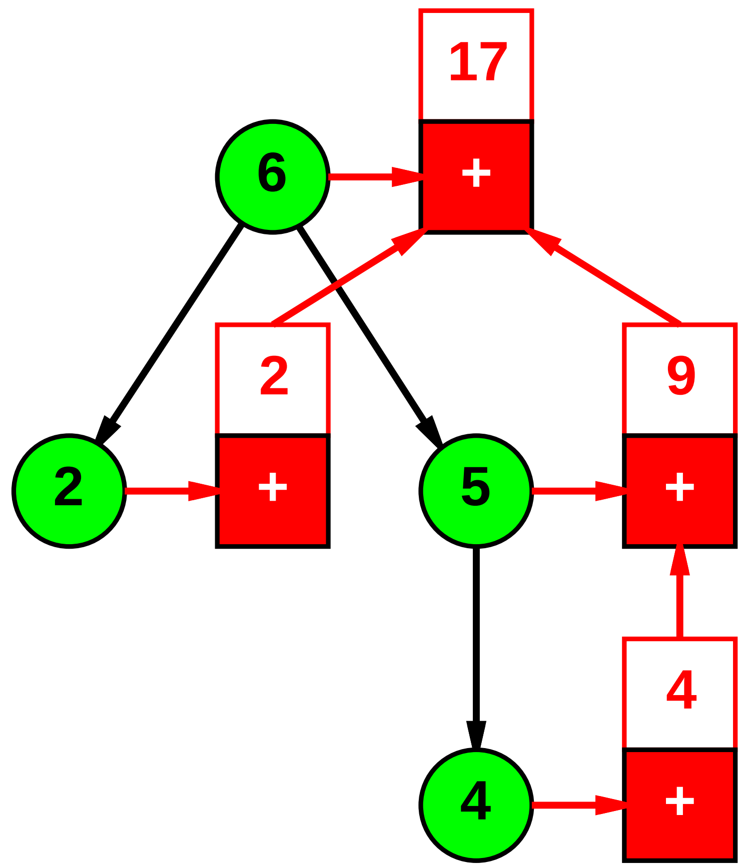
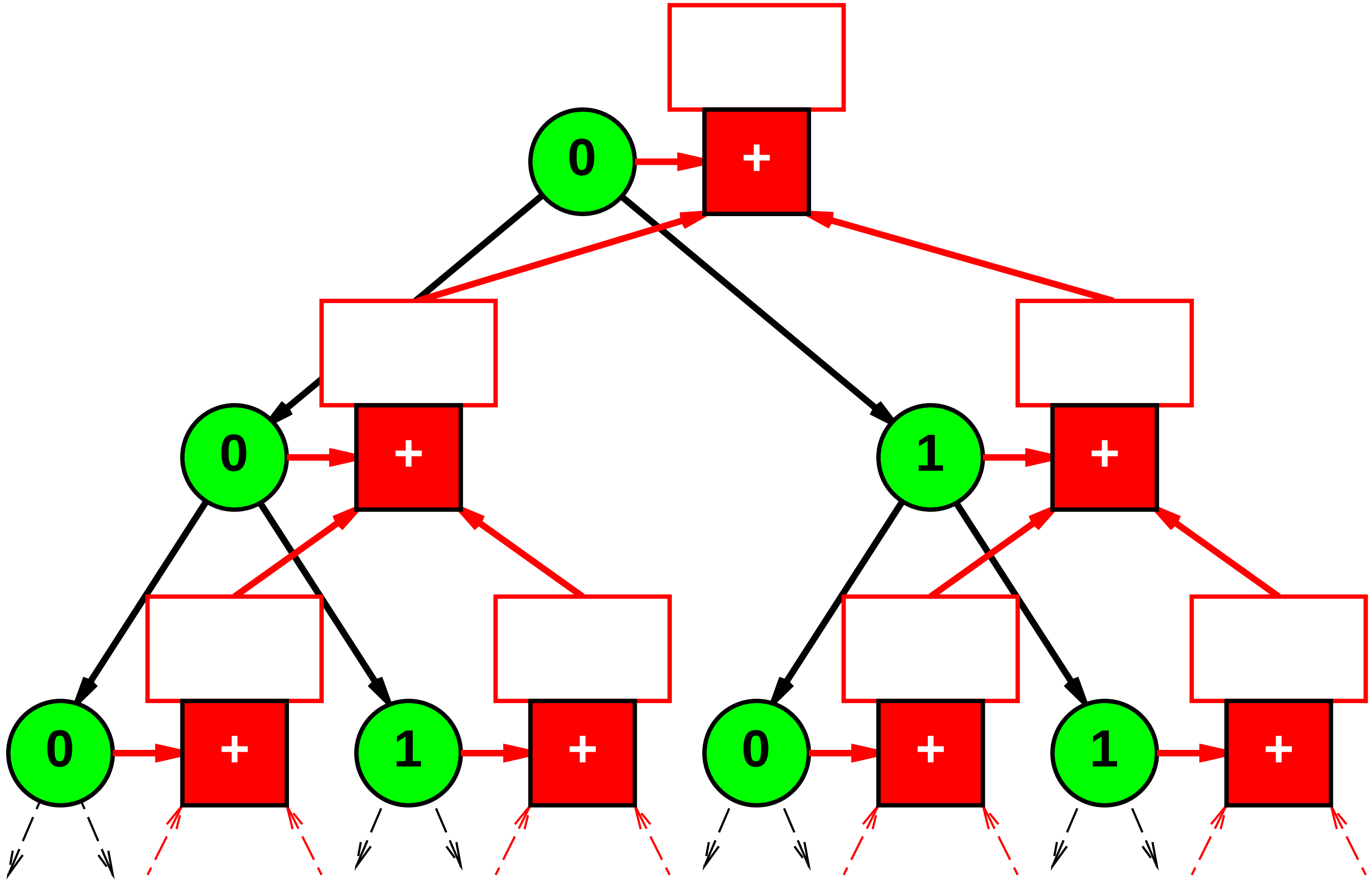
	

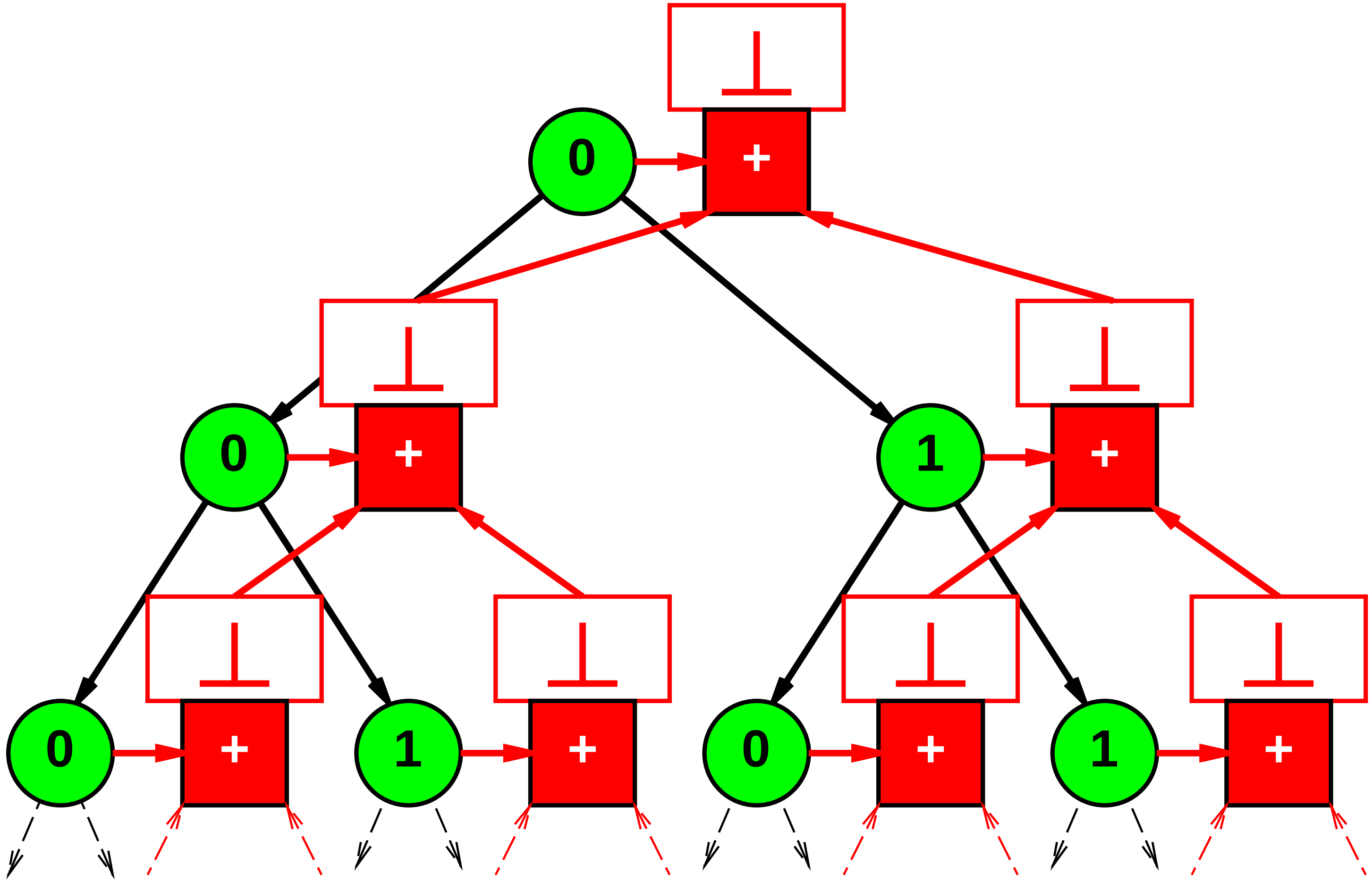
Folding over an Infinite Regular Tree



Expect θ to be ϕ-*consistent*: for each subtree t of a given tree,

$$\theta(t) = \phi(\mathsf{label}(t), \mathsf{map}(\theta)(\mathsf{children}(t))).$$

Folding over an Infinite Regular Tree



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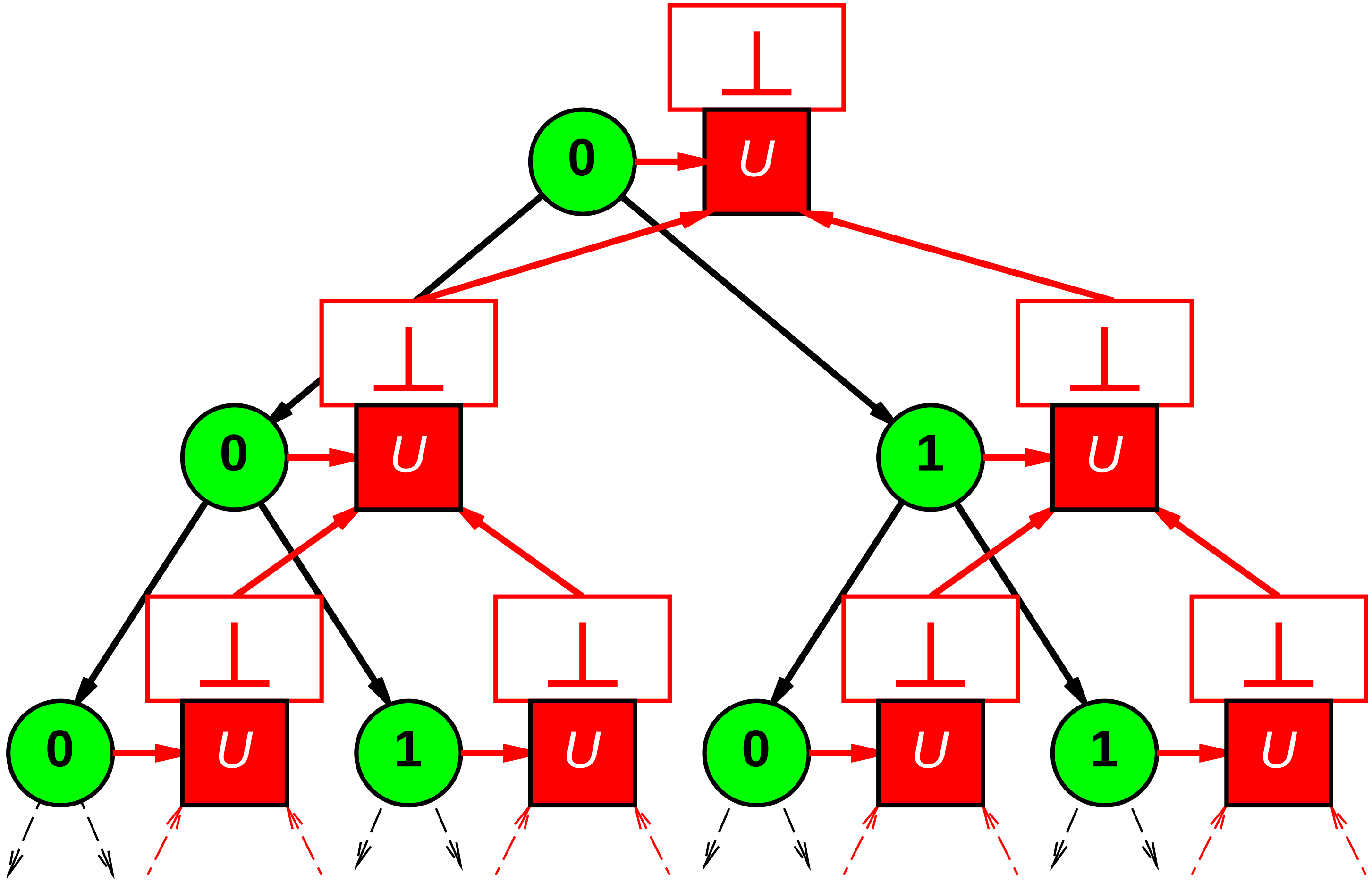

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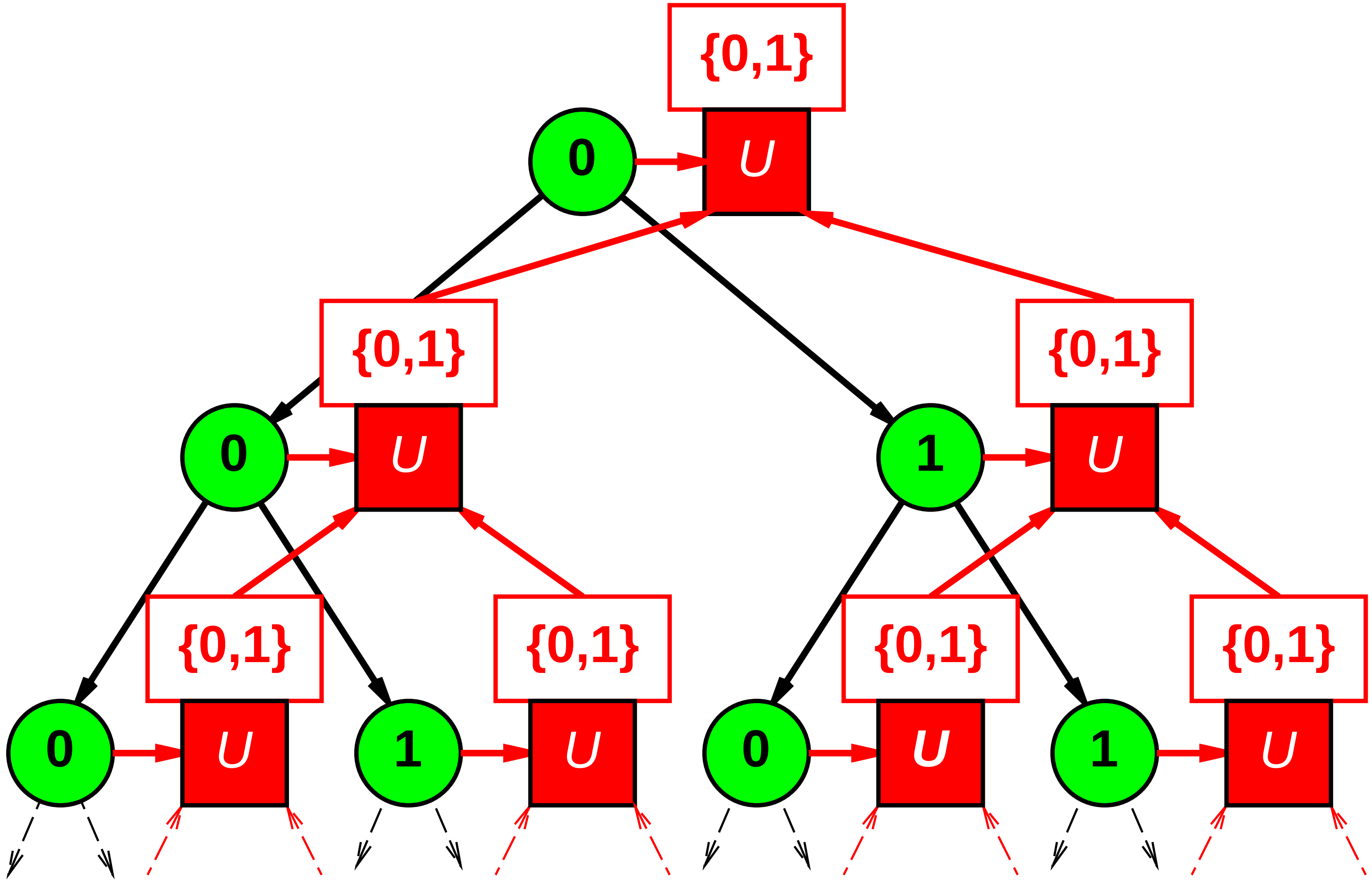
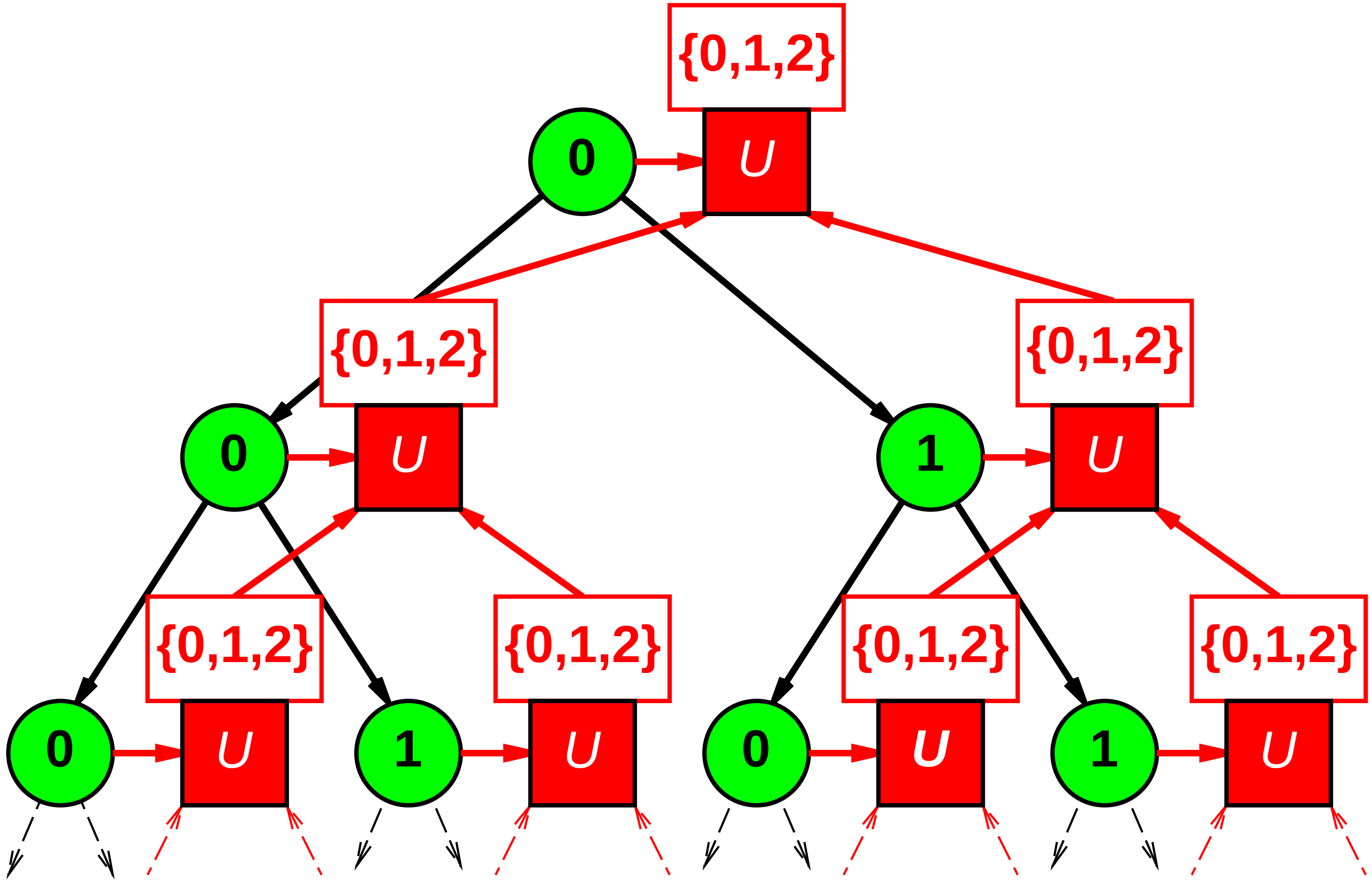


This fold may be desirable, but it is not the computed one.

Folding over an Infinite Regular Tree



This fold is not computed either.

Digression: Fixed Points

A value x is a *fixed point* of a function f if $f(x) = x$.

What are the fixed points of the following:

f_i :: Int -> Int
f_1(x) = x/2 + 3
f_2(x) = x^2
f_3(x) = x
f_4(x) = x -1

Can also have fixed points over functions manipulating data structures and other functions:

```
g :: [Int] -> [Int]
g(x) = 0:1:x

h :: (Int -> Int) -> (Int -> Int)
h(k) = \n -> if n == 0 then 1 else n*(k(n-1))
```

Digression: Least Fixed Points

Under certain conditions, functions over data structures and functions have a so-called *least fixed point*. In particular, the function must be a *continuous function* between two *pointed complete partial orders*.

- Intuitively, a pointed complete partial order is a lattice rooted at $\bot$ where elements are arranged by information content and every chain has a limit.
- Intuitively, the least fixed point of a function f is found by starting at $\bot$ and applying f until a limit is reached.
- For strict f, the least fixed point will always be $\bot$.

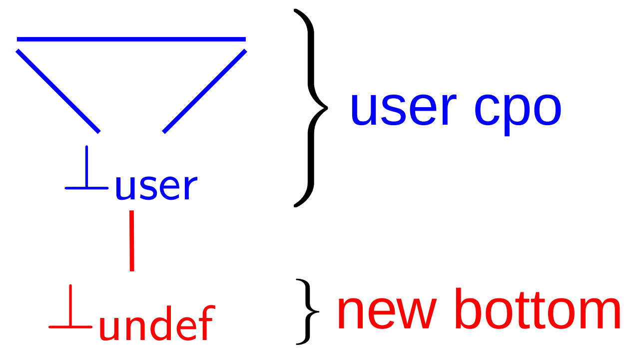
Cycfold: Goals

Given a strict combining function ϕ, want $\mathsf{cycfold}(\phi)$ that:

- Coincides with $\mathsf{fold}(\phi)$ on finite trees;
- Can return a non-trivial result for regular trees;
- Diverges on non-regular trees.

Cycfold: The Idea

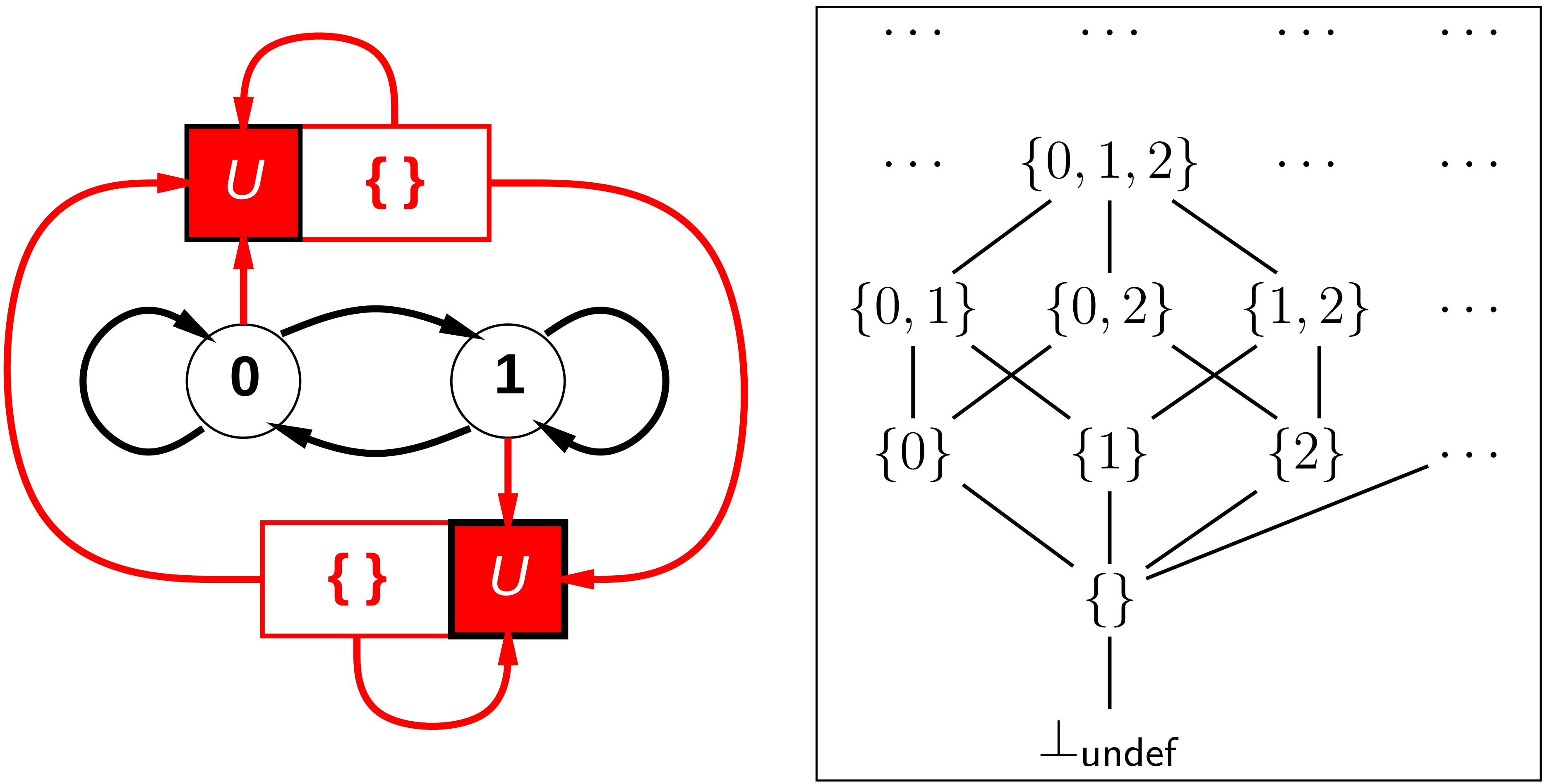
Use a result domain $\mathcal{C}_{\text{res}}$ that is a *lifted* pointed cpo (i.e., doubly pointed) and require the combining function ϕ to be strict and monotone.



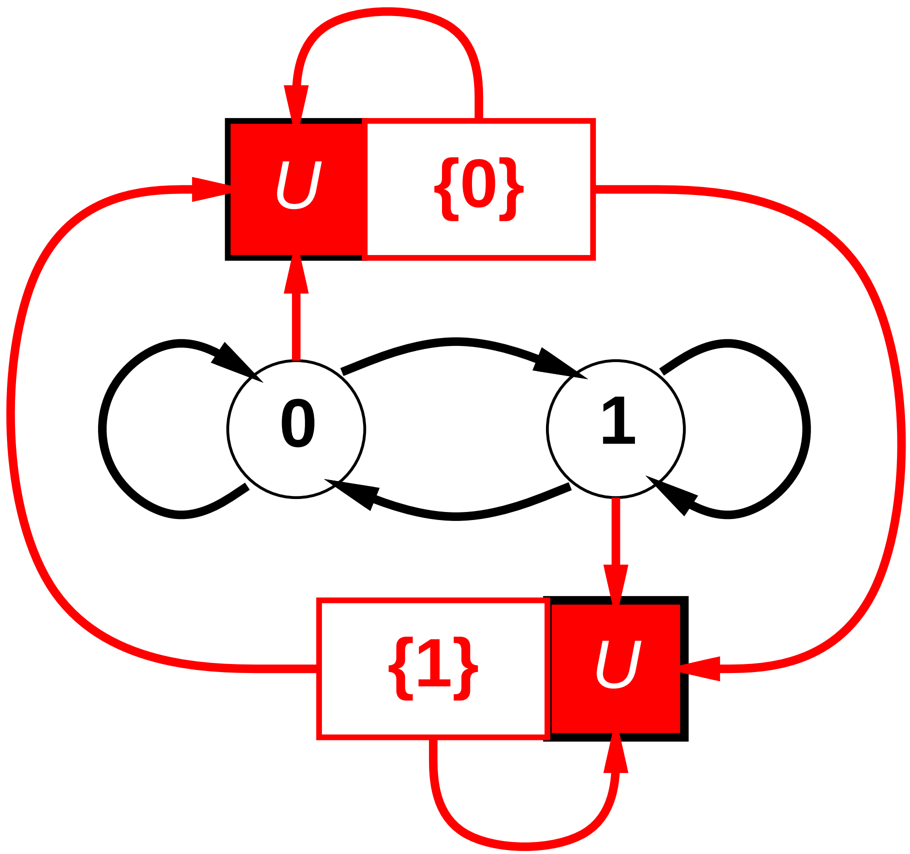
For a given tree t, calculate $\text{cycfold}(\phi)(t)$ as follows:

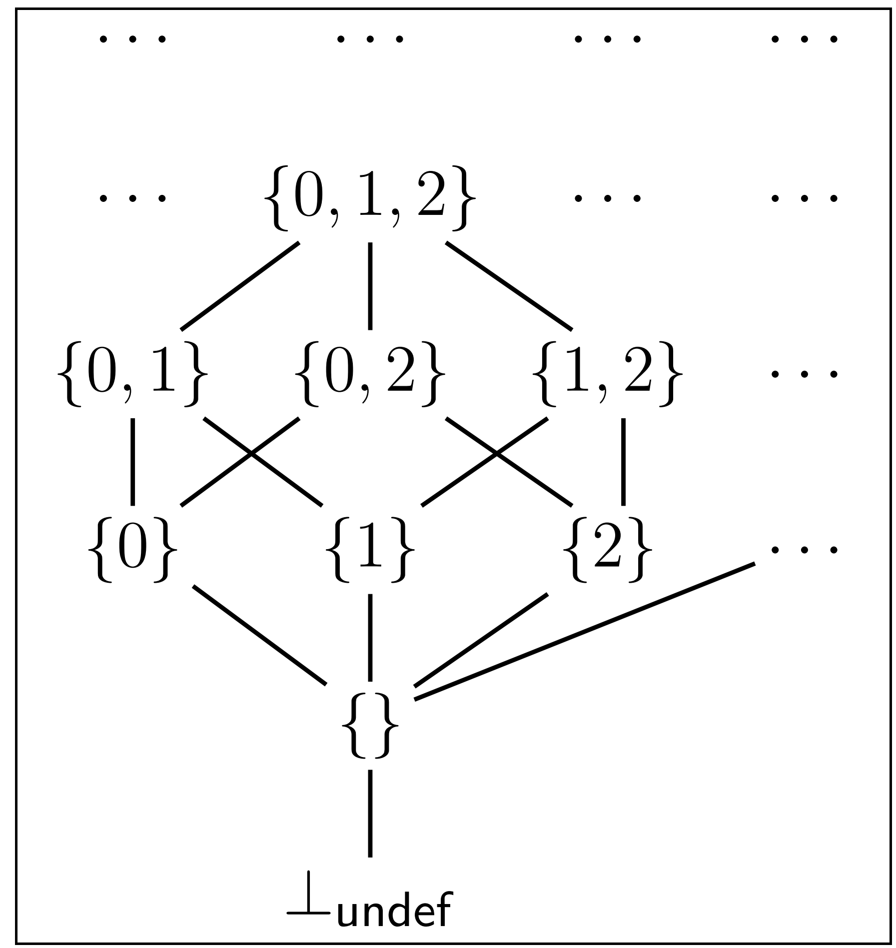
- If t not regular, return $\perp_{\text{undef}}$.
- Otherwise:
  - Let tree valuation θ_0 map all subtrees of t to $\perp_{\text{user}}$.
  - Iteratively calculate θ_{i+1} from θ_i using ϕ.
  - If $\theta_{k+1} = \theta_k$ then return $\theta_k(t)$ else return $\perp_{\text{undef}}$.

Cycfold Example 1: Node Labels

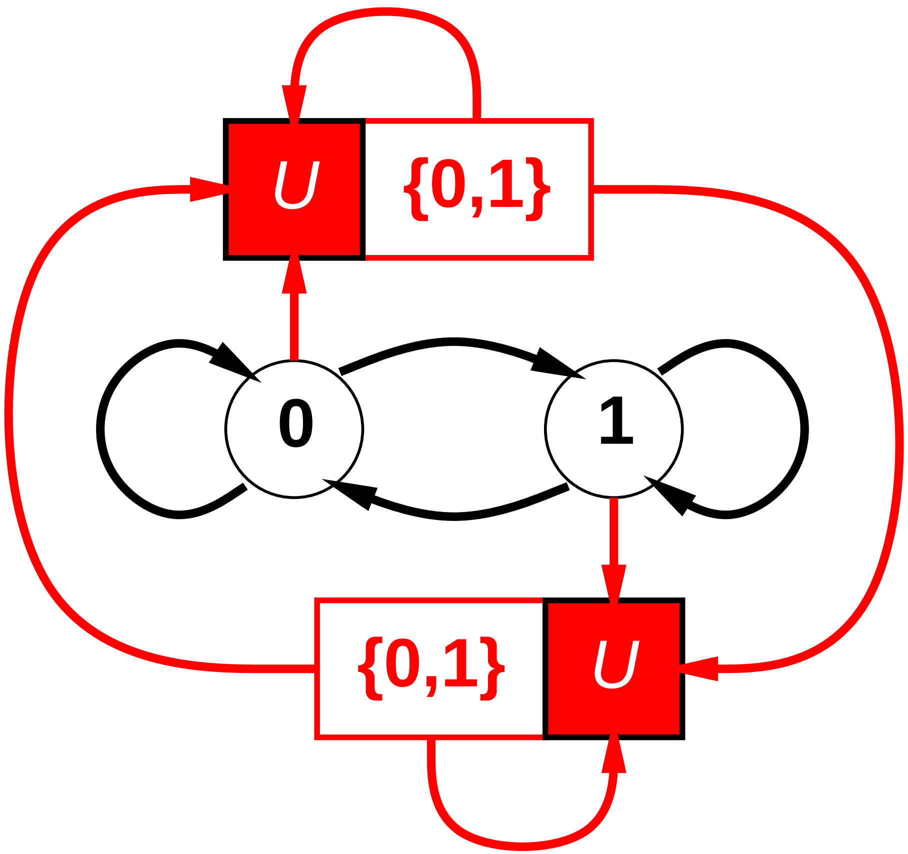


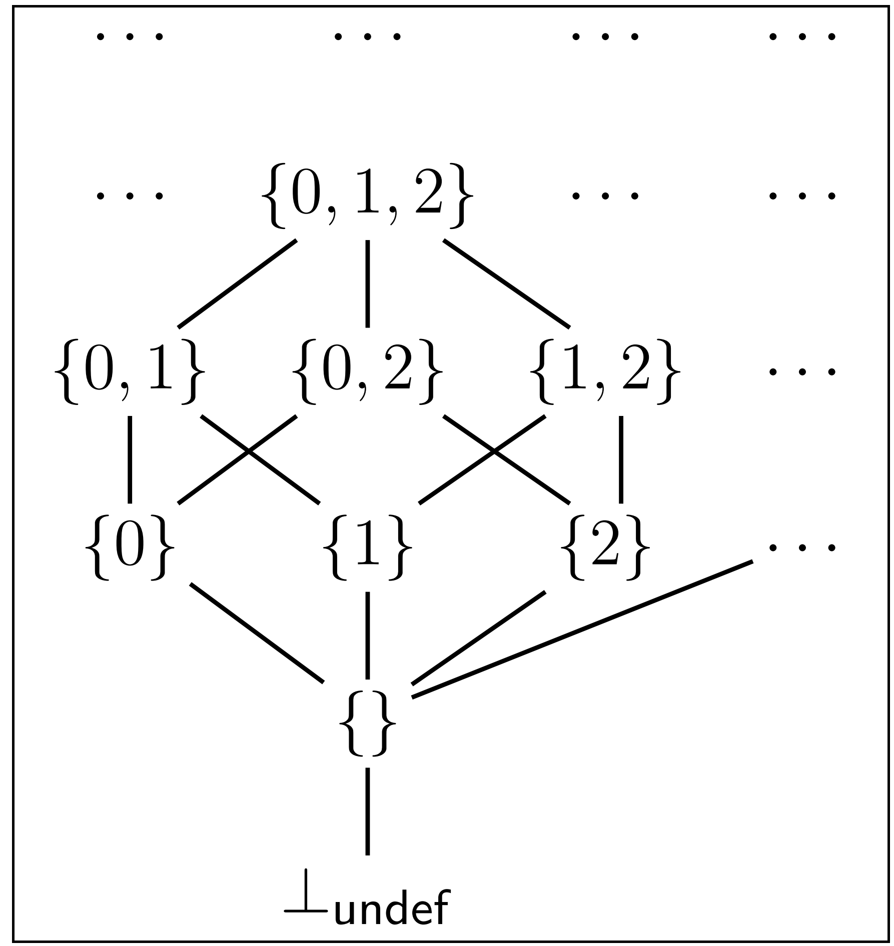
Cycfold Example 1: Node Labels



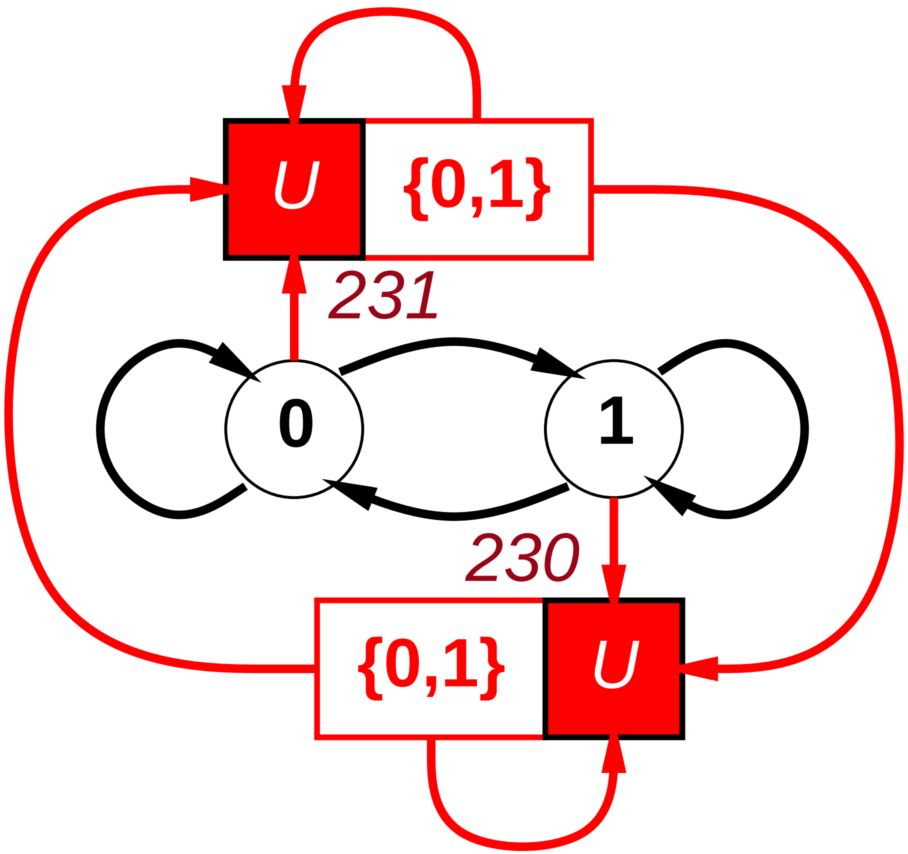


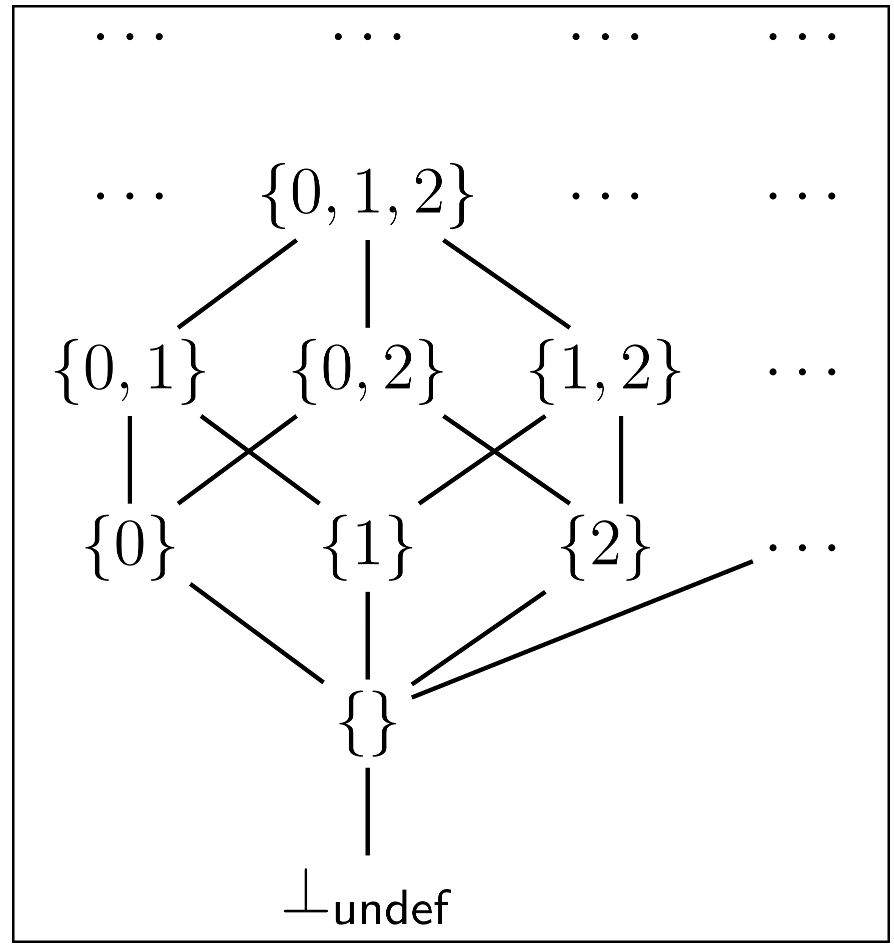
Cycfold Example 1: Node Labels





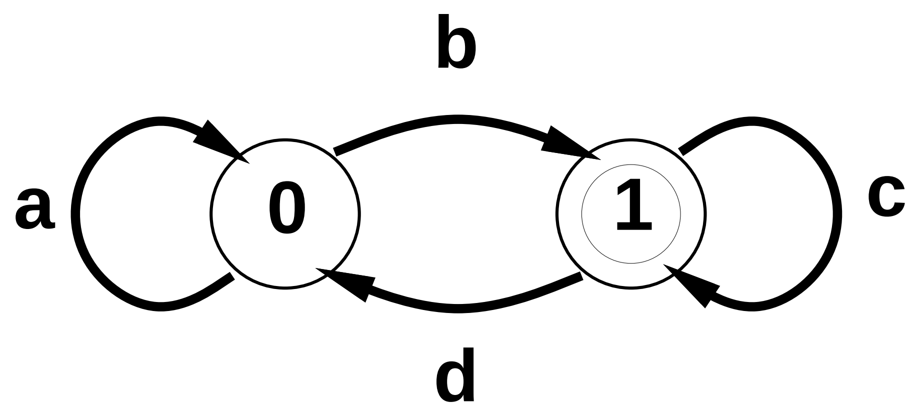
Cycfold Example 1: Node Labels

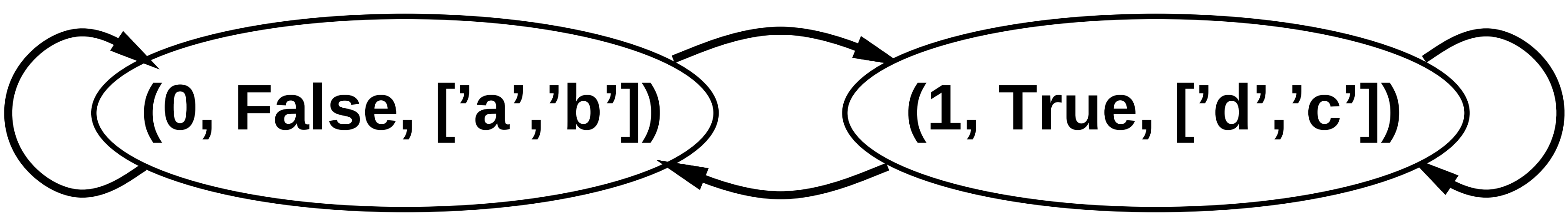




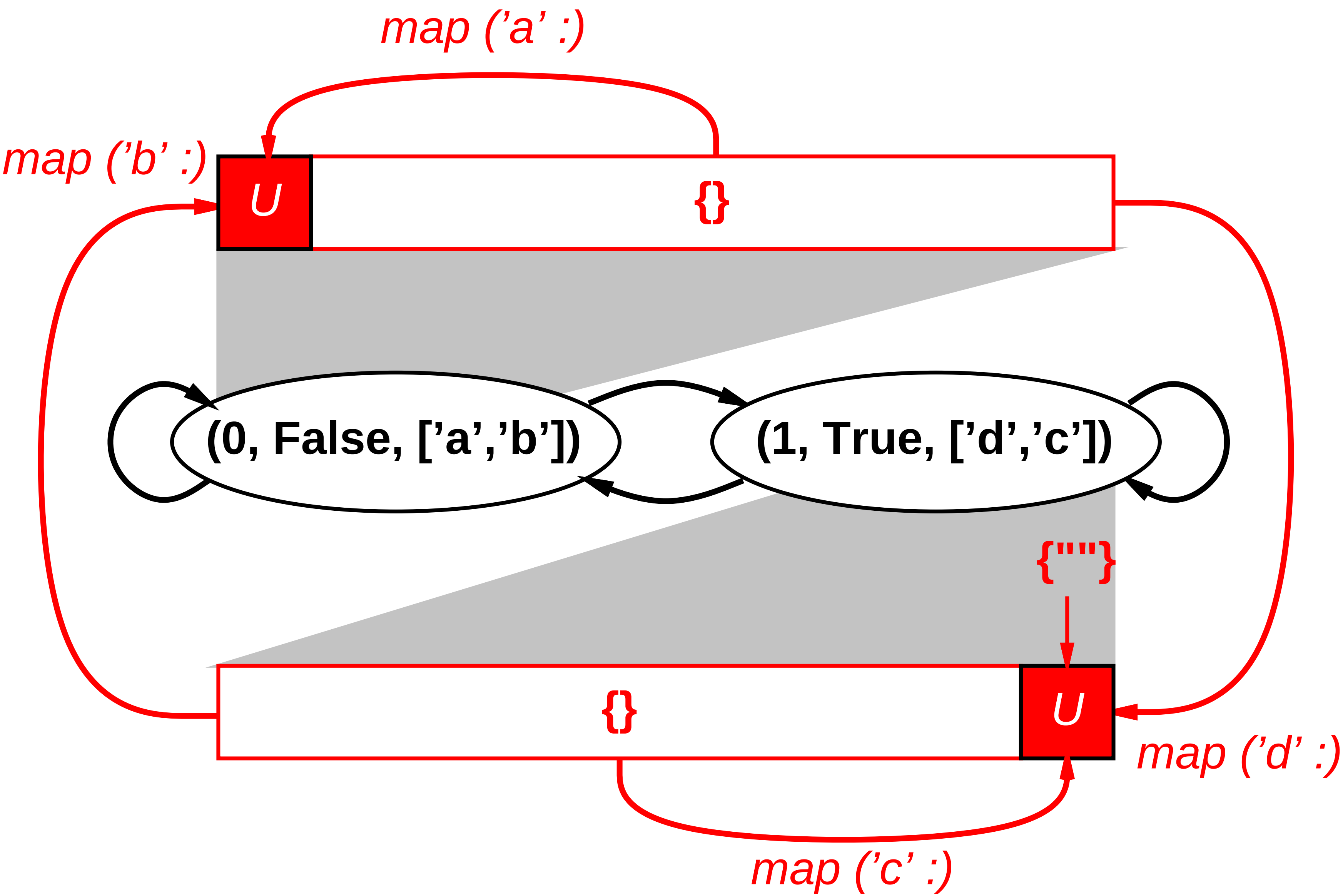
Cycfold Example 2: DFA Strings

Node labels can encode other aspects of cyclic data.

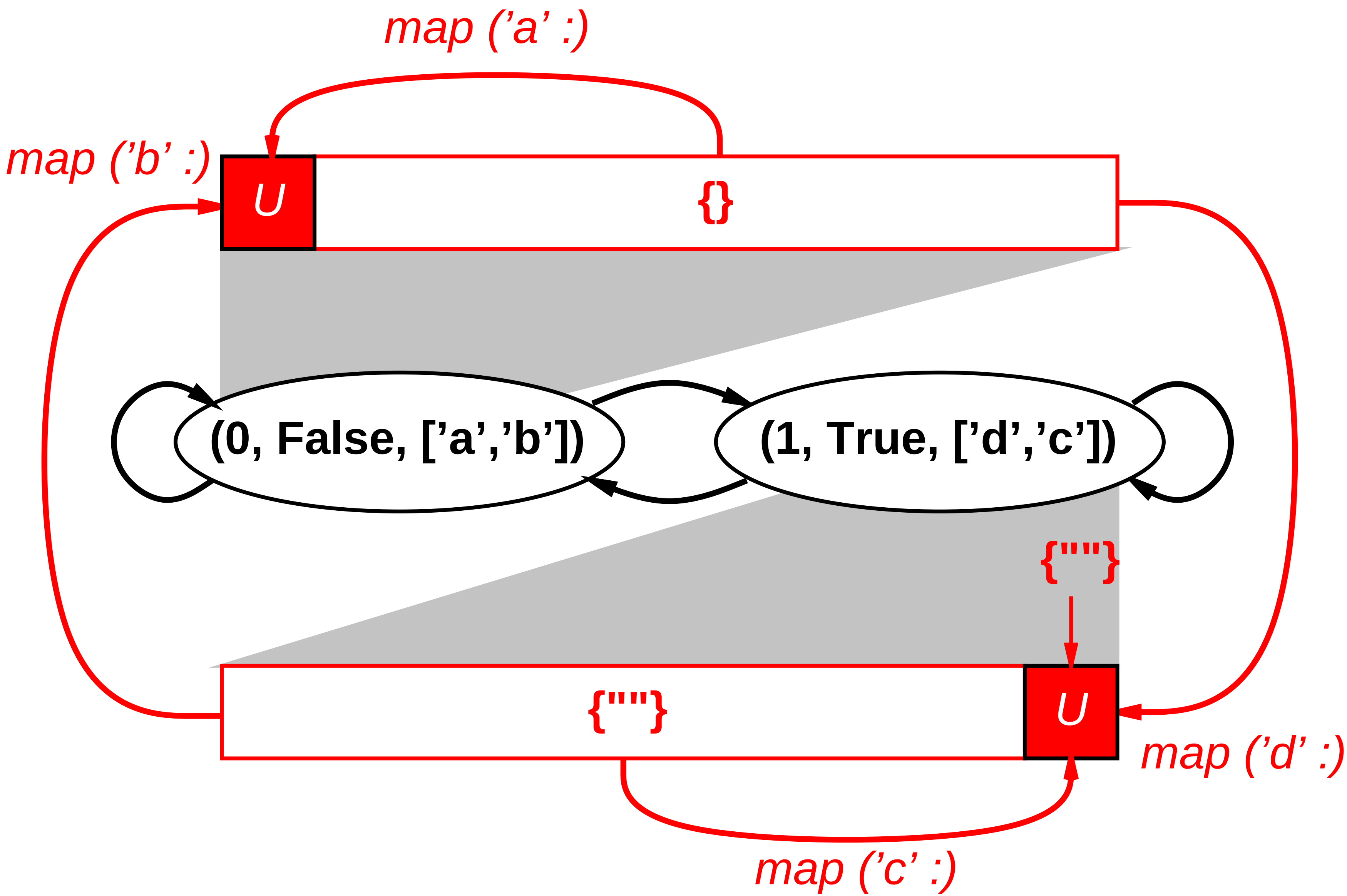




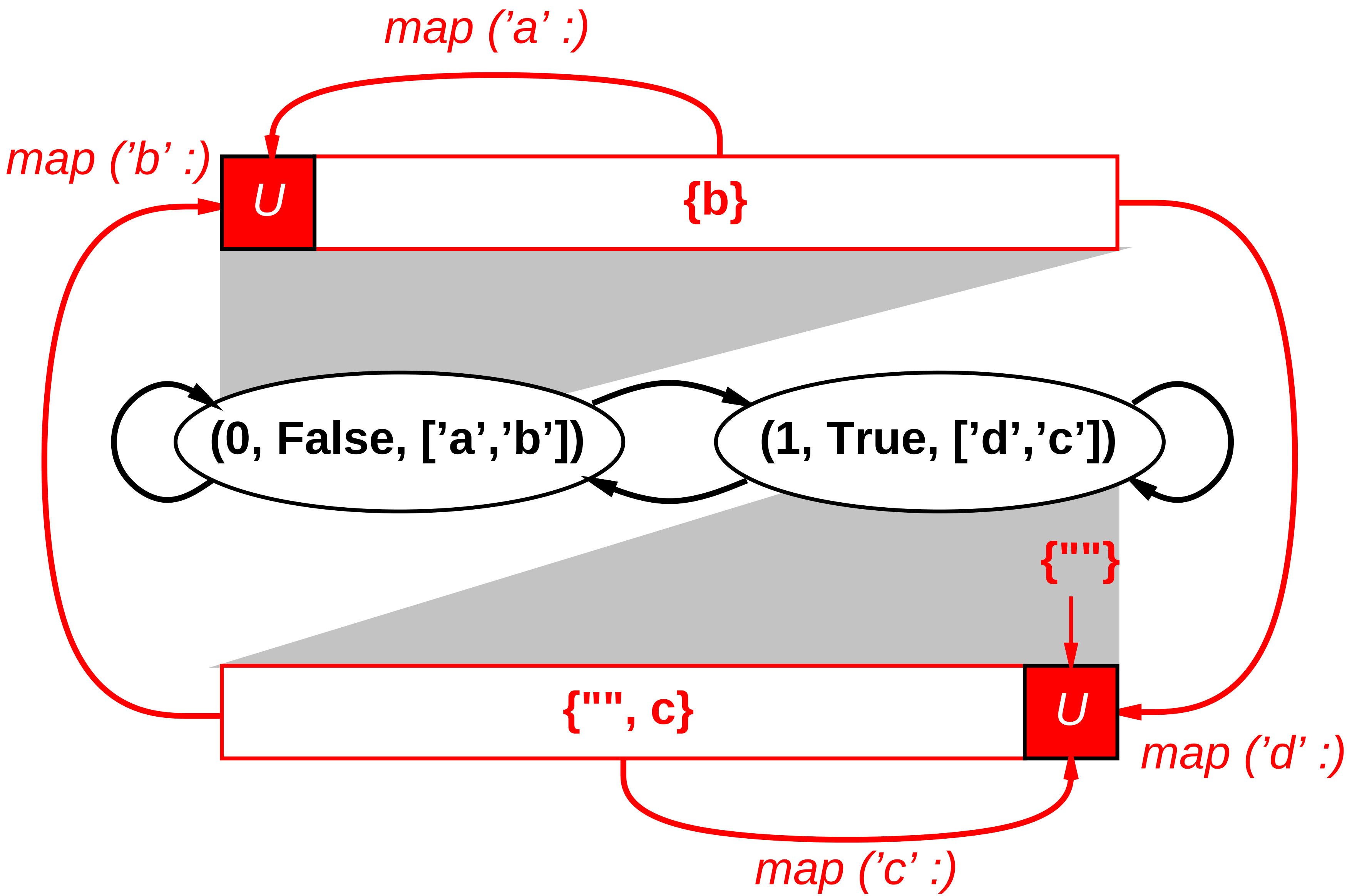
Cycfold Example 2: DFA Strings



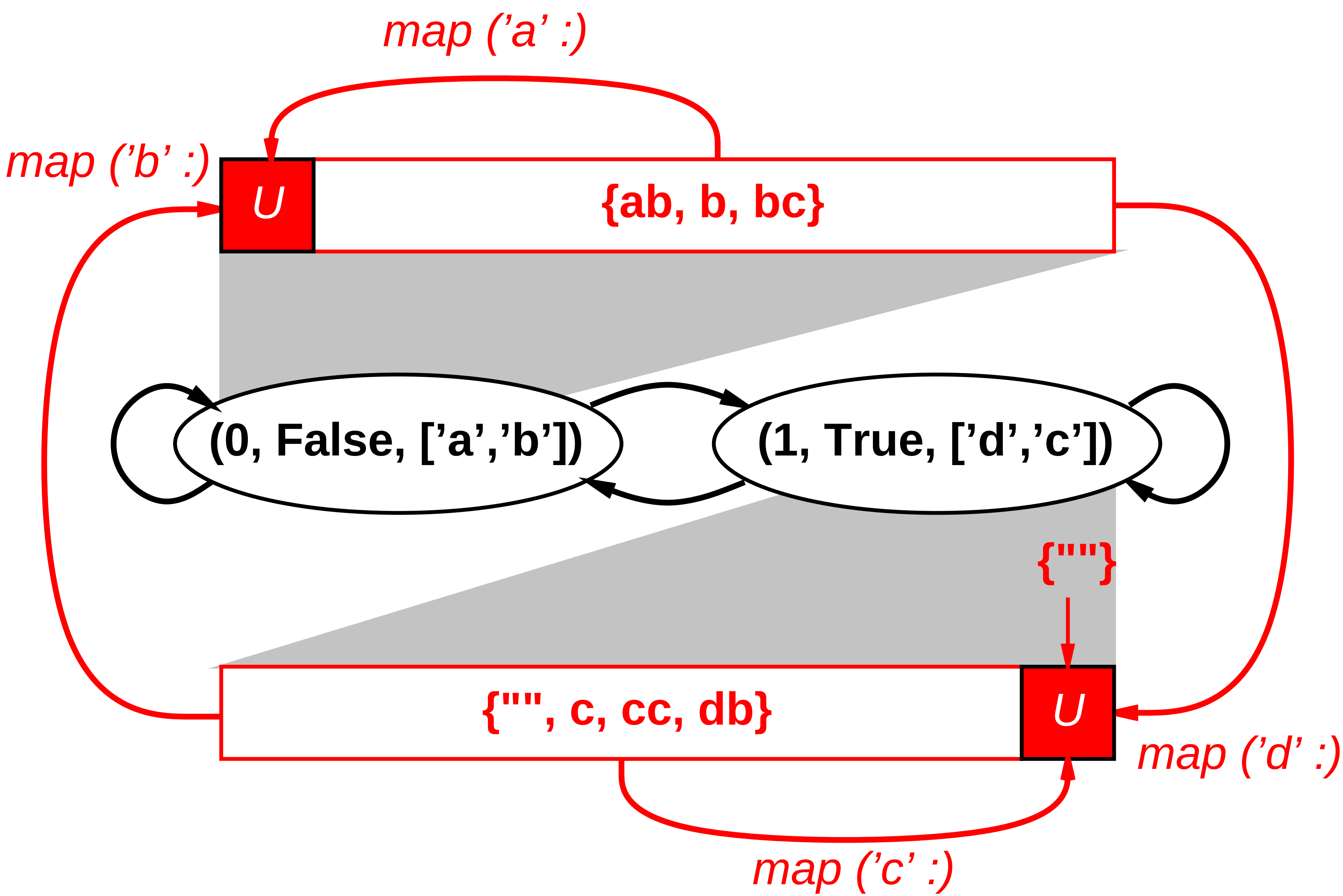
Cycfold Example 2: DFA Strings



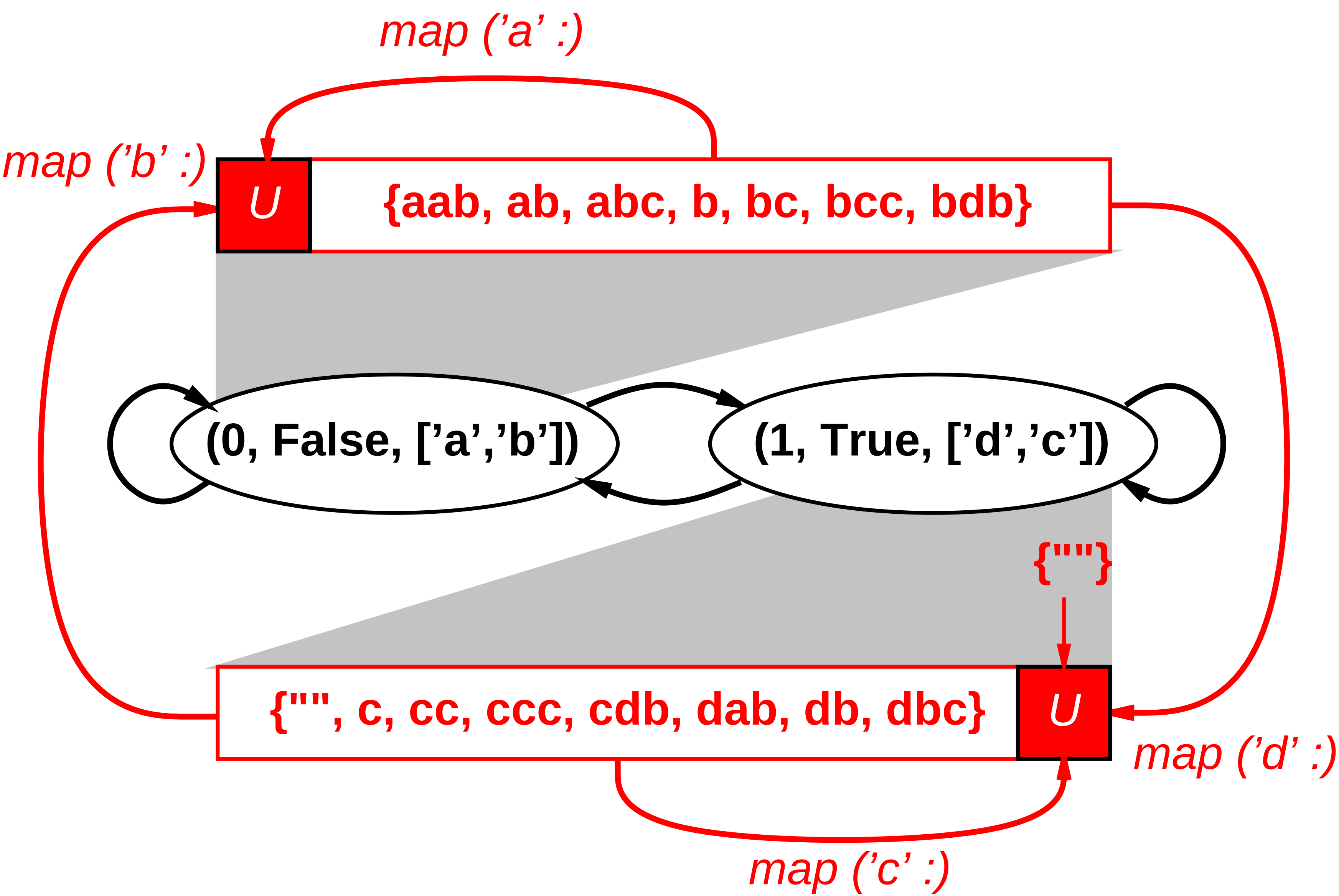
Cycfold Example 2: DFA Strings



Cycfold Example 2: DFA Strings



Cycfold Example 2: DFA Strings



Cycfold: Related Work

- Iterative fixed points common in compiler data flow.
- Graph folds (Gibbons, unpublished):
  - `ifold` = `foldtree` ∘ `untie`, analagous to fold.
  - `efold` analagous to cycfold.
- Catamorphisms over datatypes with embedded functions (Fegaras & Sheard, POPL'96):
  - Express cycles via embedded functions. E.g.,
    `val alts = Rec(fn x => Cons(0,(Cons 1 x)))`
  - Can express catamorphisms over such cycles (e.g., `map`), but these can expose the structure of the representative.

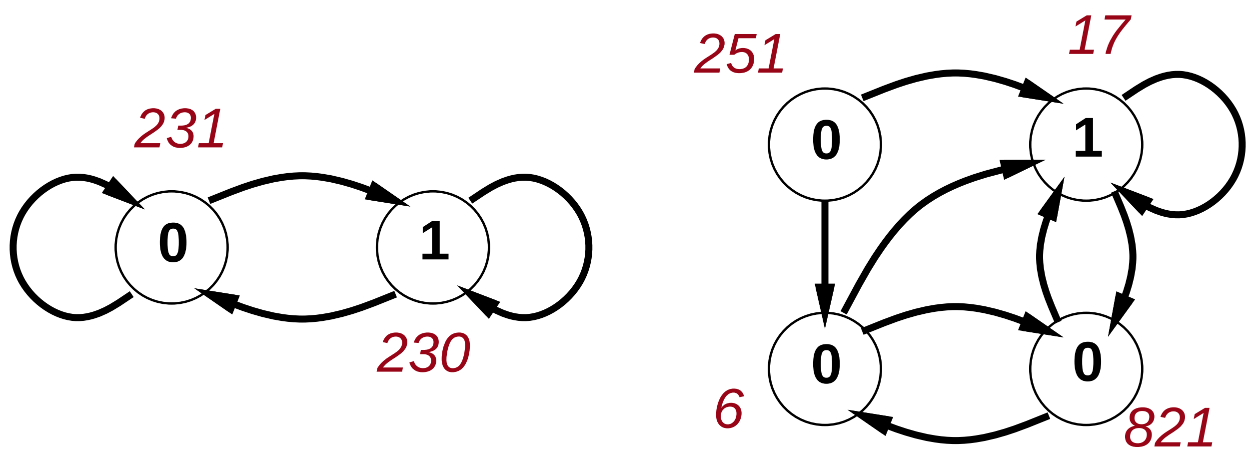
Road Map

- Viewing cyclic structures as infinite regular trees.
- Adapting the tree-generating unfold function to generate cyclic structures for infinite regular trees.
- Adapting the tree-accumulating fold function to return non-trivial results for strict combining functions and infinite regular trees.
- Cycamores: an abstraction for manipulating regular trees that we have implemented in ML and Haskell.

Cycamores

Cycamore(L) is the type of potentially cyclic graphs, with a hidden UID for each node, parameterized over label type.

- Examples:



- Key operations: make, view, unfold, fold, cycfold.
- Other operations: cycfix, memofix (see paper).
- Implementations in Standard ML and Haskell.

Cycamore Signatures 1

Standard ML		Haskell
val make :		make ::
('a * 'a Cycamore list)	*label/kids pair*	(a, [Cycamore a])
-> 'a Cycamore	*result cycamore*	-> Cycle s (Cycamore a)
val view :		view ::
'a Cycamore	*given cycamore*	Cycamore a
-> ('a * 'a Cycamore list)	*label/kids pair*	-> (a, [Cycamore a])
val unfold :		unfold ::
	order class	Ord a =>
'a MemKey	*memo key fcn.*	
-> ('a -> ('b * 'a list))	*generating fcn.*	(a -> (b, [a]))
-> 'a	*seed*	-> a
-> 'b Cycamore	*result cycamore*	-> Cycle s (Cycamore b)

Cycamore Signatures 2

Standard ML		Haskell
val fold :		fold ::
('b -> ('a list) -> 'a)	*combining fcn.*	(b -> [a] -> a)
-> ('b Cycamore)	*cycamore*	-> (Cycamore b)
-> 'a	*result*	a
val cycfold :		cycfold ::
	partial order class	(POrd a) =>
'a	*user bottom*	a
-> (('a * 'a) -> bool)	*geq*	
-> ('b -> ('a list) -> 'a)	*combining fcn.*	-> (b -> [a] -> a)
-> ('b Cycamore)	*cycamore*	-> (Cycamore b)
-> 'a	*result*	-> a

Future Work

- Theory:
  - Non-strict combining functions with cycfold.
  - Can cycfold return a cycamore?
  - Version of fold based on greatest fixed points.
- Practice:
  - Avoiding single-threaded UID generation.
  - Memoization strategies.
  - cycfold implementation heuristics.
  - Cyclic hash-consing experimentation.
- Extending ML/Haskell with general cyclic data types.