CS230 Data Structures
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Handout # 35
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2-3 Trees

Balanced Search Trees

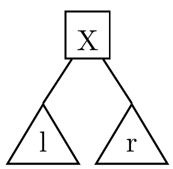
Many data structures use binary search trees or generalizations thereof. Operations on such search trees are often proportional to the height of the tree. To guarantee that such operations are efficient, it is necessary to ensure that the height of the tree is logarithmic in the number of nodes. This is usually achieved by some sort of **balancing** mechanism that guarantees that subtrees of a node never differ "too much" in their heights (by either an additive or multiplicative factor).

There are many kinds of balanced search trees. Here we study a particularly elegant form of balanced search tree known as a **2-3 tree**. There are many other kinds of balanced search trees (e.g., red-black trees, AVL trees, 2-3-4 trees, and B-trees), some of which you will encounter in CS231.

2-3 Trees

A 2-3 tree has three different kinds of nodes:

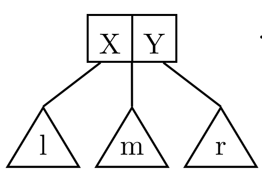
1. A leaf, written as ●.
2. A 2-node, written as

   X is called the **value** of the 2-node; l is its **left subtree**; and r is its **right subtree**. Every 2-node must satisfy the following invariants:

   (a) Every value v appearing in subtree l must be $\leq X$.
   (b) Every value v appearing in subtree r must be $\geq X$.
   (c) The length of the path from the 2-node to *every* leaf in its subtrees must be the same.

3. A 3-node, written as

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   X is called the **left value** of the 3-node; Y is called the **right value** of the 3-node; l is its **left subtree**; m is its **middle subtree**; and r is its **right subtree**.

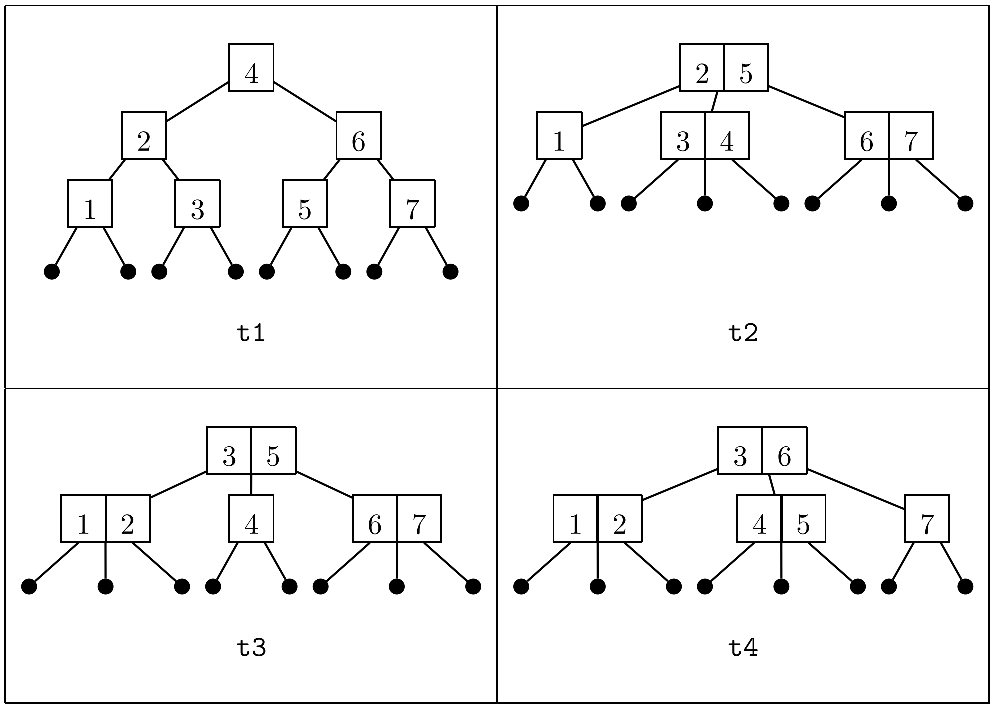
   Every 3-node must satisfy the following invariants:

   (a) Every value v appearing in subtree l must be $\leq X$.
   (b) Every value v appearing in subtree m must be $\geq X$ and $\leq Y$.
   (c) Every value v appearing in subtree r must be $\geq Y$.
   (d) The length of a path from the 3-node to *every* leaf in its subtrees must be the same.

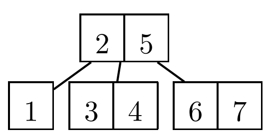
The last invariant for 2-nodes and 3-nodes is the **path-length invariant**. The balance of 2-3 trees is a consequence of this invariant. The height of a 2-3 tree with n nodes cannot exceed $\log_2(n+1)$. Together, the tree balance and the ordered nature of the nodes means that testing membership in, inserting an element into, and deleting an element from a 2-3 tree takes logarithmic time.

2-3 Tree Examples

Given a collection of three or more values, there are several 2-3 trees containing those values. For instance, below are all four distinct 2-3 trees containing first 7 positive integers.



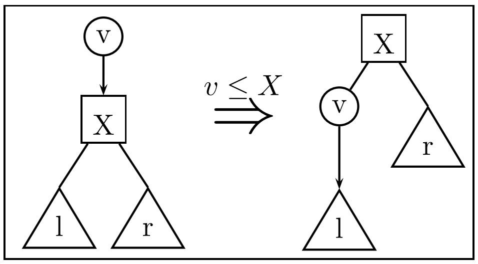
We shall use the term **terminal node** to refer to a node that has leaves as its subtrees. To save space, we often will not explicitly show the leaves that are the childred of a terminal node. For instance, here is another depiction of the tree `t2` above without the explicit leaves:

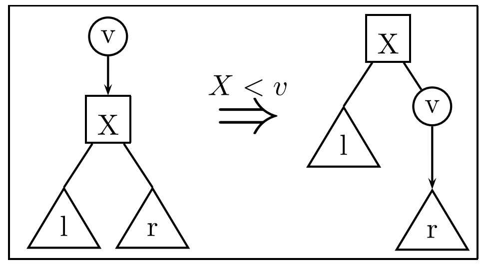


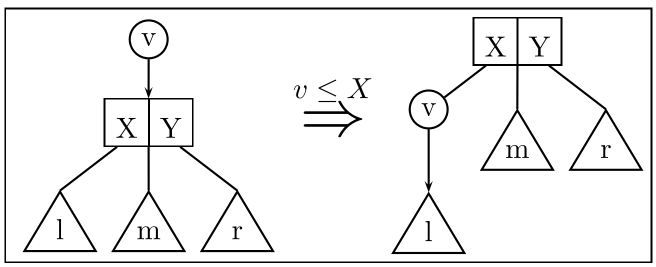
2-3 Tree Insertion: Downward Phase

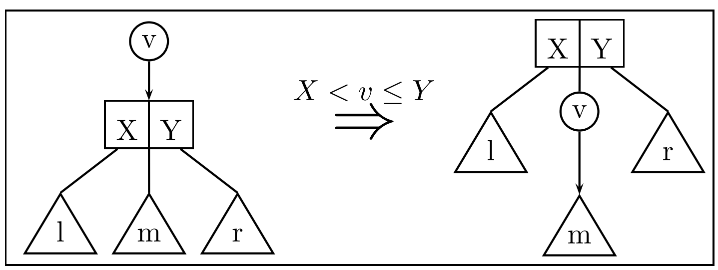
When inserting an element v into a 2-3 tree, care is required to maintain the invariants of 2-nodes and 3-nodes. As shown in the rules below, the order invariants are maintained much as in a binary search tree by comparing v to the node values encountered when descending the tree and moving in a direction that satisfies the order invariants.

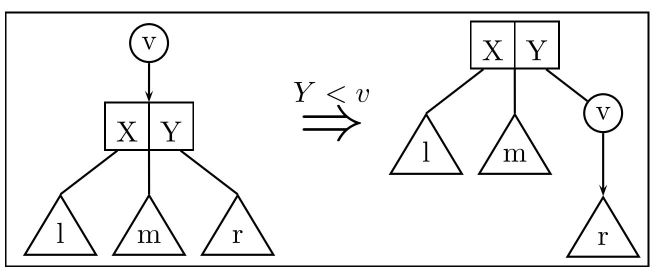
In the following rules, the result of inserting an element v into a 2-3 tree is depicted as a circled v with an arrow pointing down toward the tree in which it is to be inserted. X and Y are variables that stand for any elements, while triangles labeled l, m, and r stand for whole subtrees.







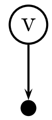




The rules state that elements equal to a node value are always inserted to the left of the node. This is completely arbitrary; they could be inserted to the right as well.

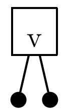
Note that the tree that results from inserting v into a tree T had better not have a different height from T. Otherwise, the path-length invariant would be violated. We will see how this plays out below.

2-3 Tree Insertion: Base Case

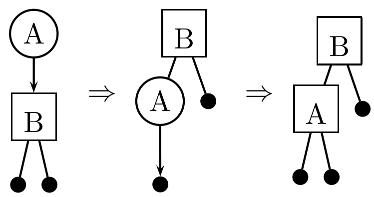
What happens when v reaches a leaf? 

It's impossible to push it down further, and it's impossible to "absorb" it.

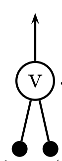
Intuitively, our only option is to make a 2-node whose value is v with two leaf subtrees:

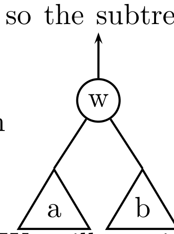


But then the path-length invariant would be violated, since the height would increase from 0 to 1. For example:



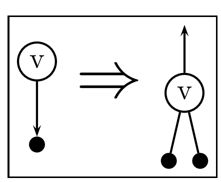
The rightmost tree above is *not* a valid 2-3 tree because different paths from B to a leaf have different lengths.

Instead of making a "real" 2-node, we'll make a "pseudo" 2-node depicted as 

. This notation indicates that there is "no room" for the element v "downstairs", so it is being "kicked upstairs". As we shall see, a value may need to be kicked up many levels, so the subtrees of a kicked-up node can be non-trivial. In general, a kicked-up node has the form 

where w is the kicked-up value and a and b are 2-3 trees *with the same height* h. We will consider the "height" of such a configuration to be the height of its subtrees. This is because w is not really considered a part of the configuration, and so does not contribute to its height.

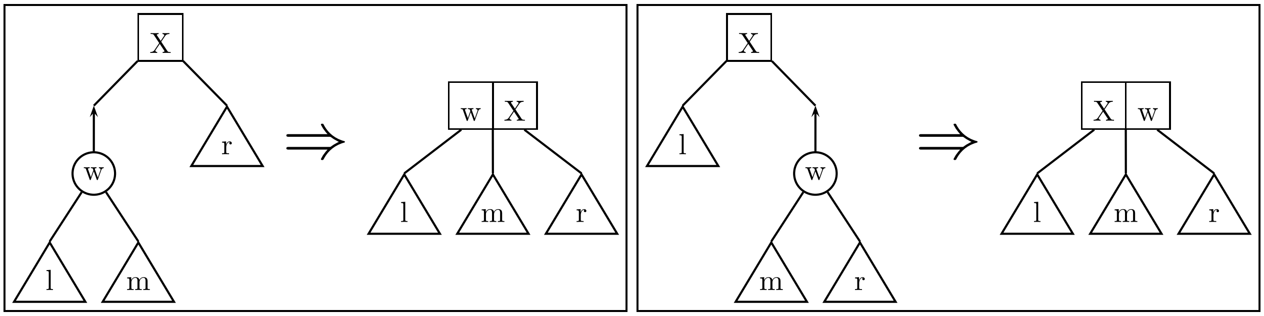
So we are left with the following rule for insertion into a leaf:



This base case converts the downward phase of insertion into an upward phase. The rest of the insertion rules now deal with the upward phase.

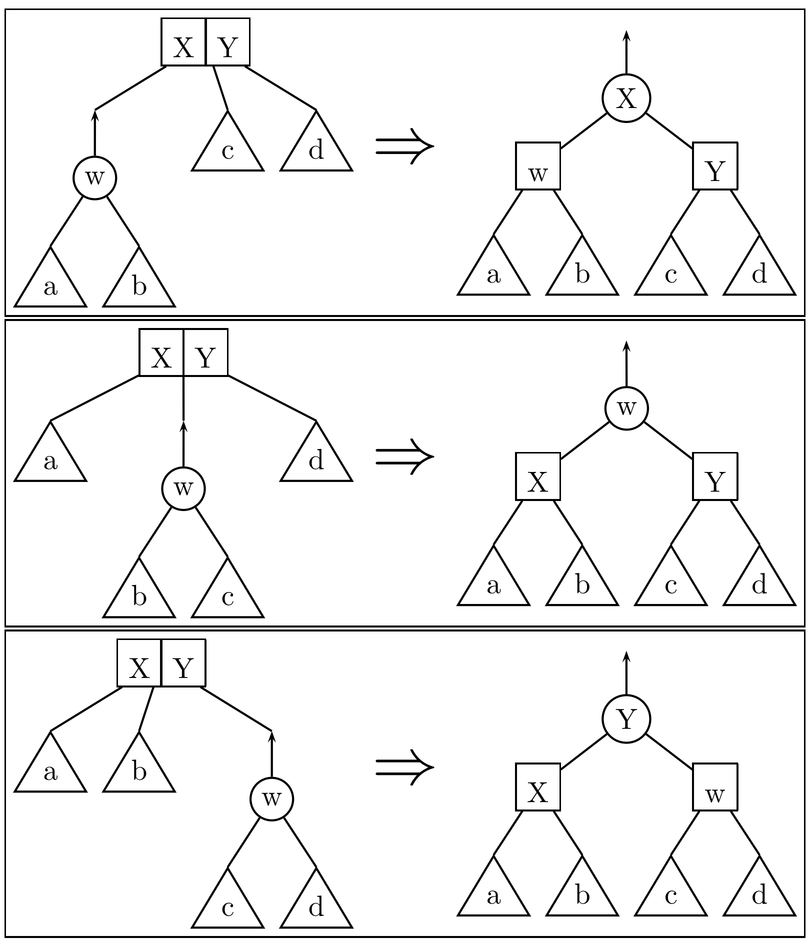
2-3 Tree Insertion: Upward Phase

If there is a 2-node upstairs, the kicked-up value w can be absorbed by the 2-node:



By our assumptions about height, the resulting tree is a valid 2-3 tree.

If there is a 3-node upstairs, w cannot simply be absorbed. Instead, the 3-node is split into two 2-nodes that become the subtrees of a new kicked-up node one level higher. The value w and the two 3-node values X and Y are appropriately redistributed so that the middle of the three values is kicked upstairs at the higher level:

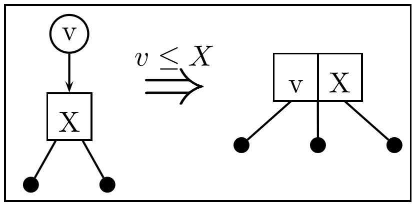


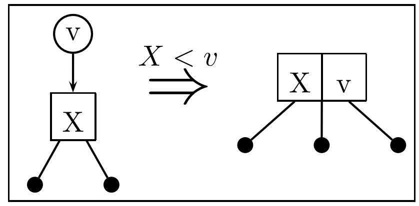
The kicking-up process continues until either the kicked-up value is absorbed or the root of the tree is reached. In the latter case, the kicked-up value becomes the value of a new 2-node that increases the height of the tree by one. This is the only way that the height of a 2-3 tree can increase.

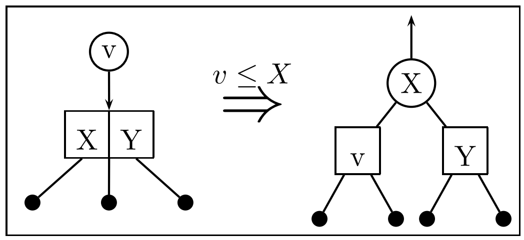
Convince yourself that heights and element order are unchanged by the downward or upward phases of the insertion algorithm. This means that the tree result from insertion is a valid 2-3 tree.

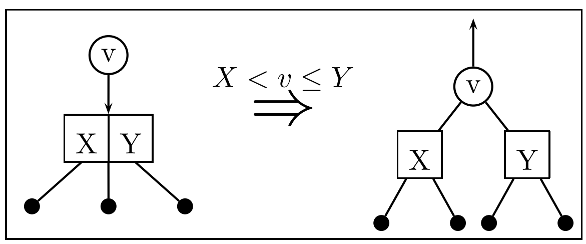
2-3 Tree Insertion: Special Cases for Terminal Nodes

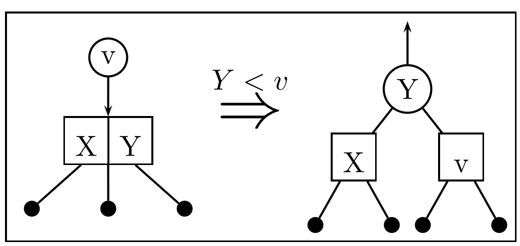
The aforementioned rules are all the rules needed for insertion. However, insertion into terminal nodes is tedious because the inserted value will be pushed down to a leaf and then reflected up right away. To reduce the number of steps performed in examples, we can pretend that insertion into terminal nodes is handled by the following rules:











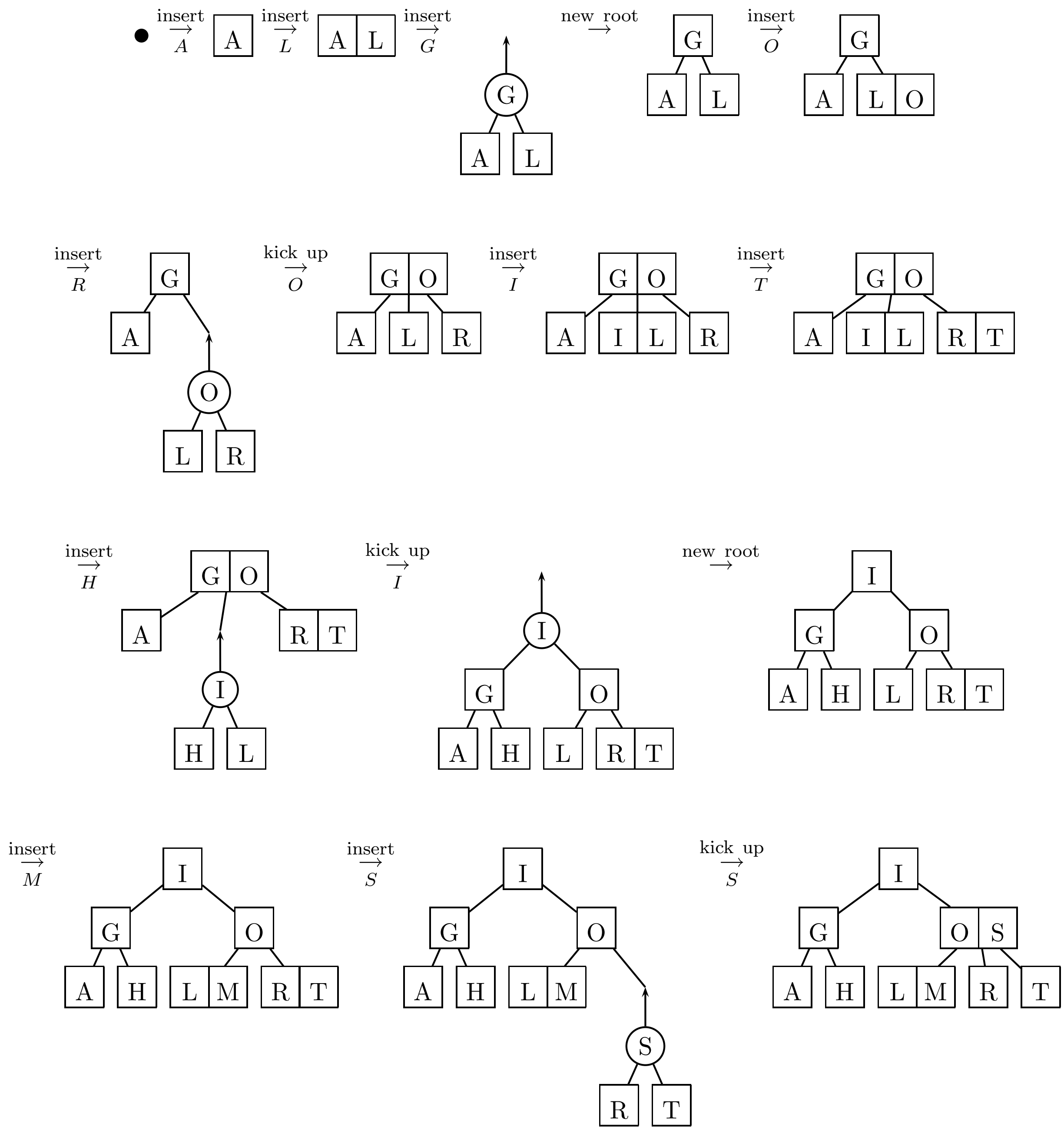
2-3 Tree Insertion: Implementation

The above rules can be implemented in any programming language, though the implementation is significantly easier in languages that support so-called **algebraic** or **sum-of-products** datatypes (such as OCaml and Haskell). Insertion can be implemented as a recursive method/procedure/function that returns either a regular 2-3 tree or a "kicked-up" configuration. The upward phase rules are implemented by handling a "kicked-up" configuration result from a recursive subcall.

We haven't said whether our 2-3-trees are immutable or mutable. It turns out it doesn't matter – the above insertion rules work for either case if interpreted appropriately.

2-3 Tree Insertion Example

Here we show the letter-by-letter insertion of the letters `A L G O R I T H M S` into an initially empty 2-3 tree. We use the terminal node optimizations to simplify the sequence.



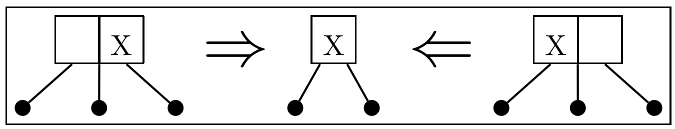
2-3 Tree Deletion: Downward Phase

The downward phase for deleting an element from a 2-3 tree is the same as the downward phase for inserting an element *except* for the case when the element to be deleted is equal to the value in a 2-node or a 3-node. In this case, if the value is not part of a terminal node, the value is replaced by its in-order predecessor or in-order successor, just as in binary search tree deletion. So in any case, deletion leaves a **hole** in a terminal node.

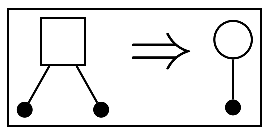
The goal of the rest of the deletion algorithm is to remove the hole without violating the other invariants of the 2-3 tree.

2-3 Tree Deletion: Terminal Cases

Handling the removal of a hole from a terminal 3-node is easy: just turn it into a 2-node!



To deal with a hole in a terminal 2-node, we consider it to be a special **hole node** that has a *single* subtree.



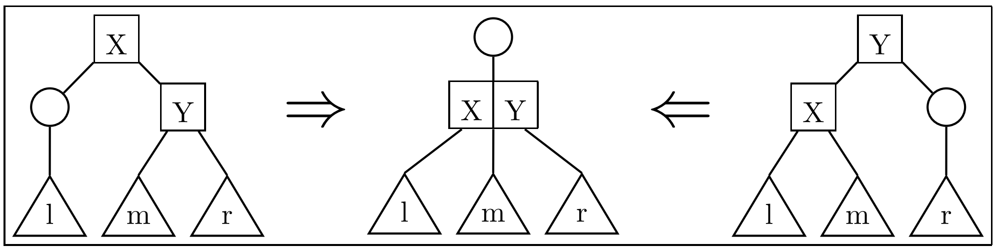
For the purposes of calculating heights, such a hole node *does* contribute to the height of the tree. This decision allows the the path-length invariant to be preserved in trees with holes.

2-3 Tree Deletion: Upward Phase

The goal of the upward phase of 2-3 tree deletion is to propagate the hole up the tree until it can be eliminated. It is eliminated either (1) by being "absorbed" into the tree (as in the cases 2, 3, and 4 below) or (2) by being propagated all the way to the root of the 2-3 tree by repeated applications of the case 1. If a hole node propagates all the way to the top of a tree, it is simply removed, decreasing the height of the 2-3 tree by one. This is the only way that the height of a 2-3 node can decrease.

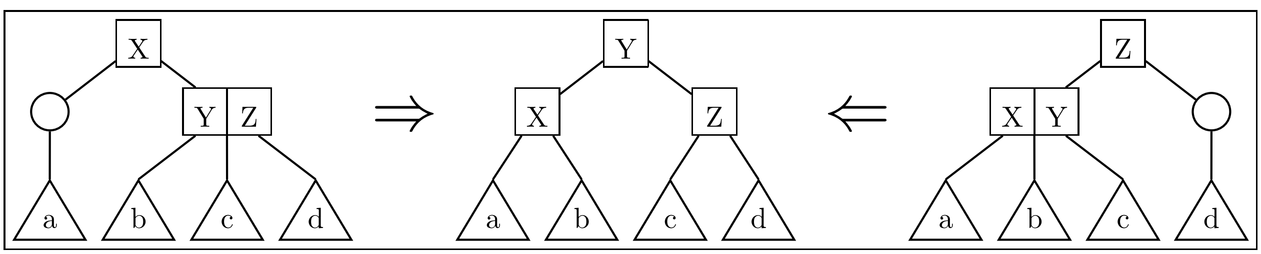
There are four cases for hole propagation/removal, which are detailed below. You should convince yourself that each rule preserves both the element-order and path-length invariants.

1. *The hole has a 2-node as a parent and a 2-node as a sibling.*



In this case, the heights of the subtrees l, m, and r are the same.

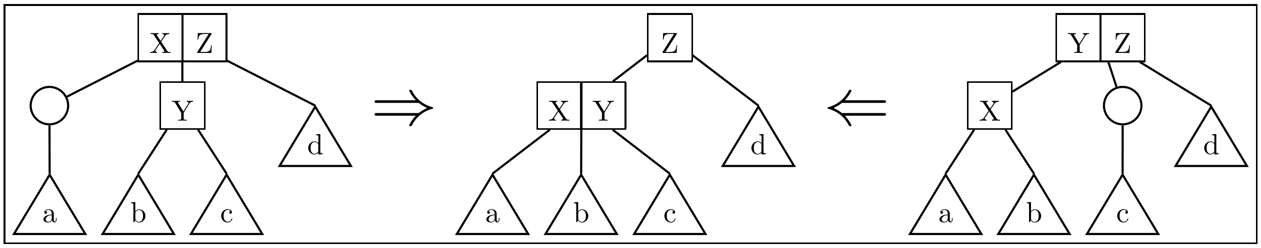
2. *The hole has a 2-node as a parent and a 3-node as a sibling.*



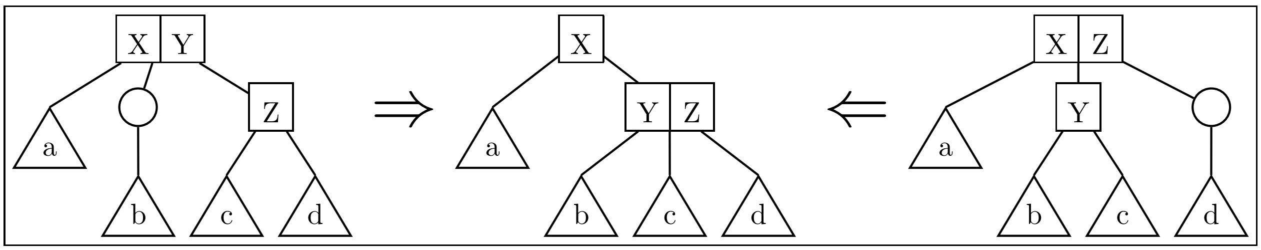
In this case, the heights of the subtrees a, b, c, and d are the same.

3. *The hole has a 3-node as a parent and a 2-node as a sibling.* There are two subcases:

   (a) The first subcase involves subtrees a, b, and c whose heights are one less than that of subtree d.

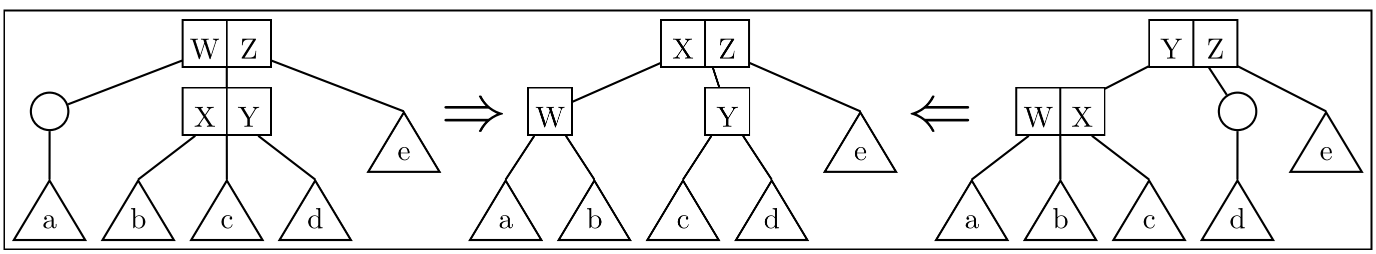


   (b) The second subcase involves subtrees b, c, and d whose heights are one less than that of subtree a. When the hole is in the middle, there may be ambiguity in terms of whether to apply the right-hand rule of the first subcase or the left-hand rule of the second subcase. Either application is OK.

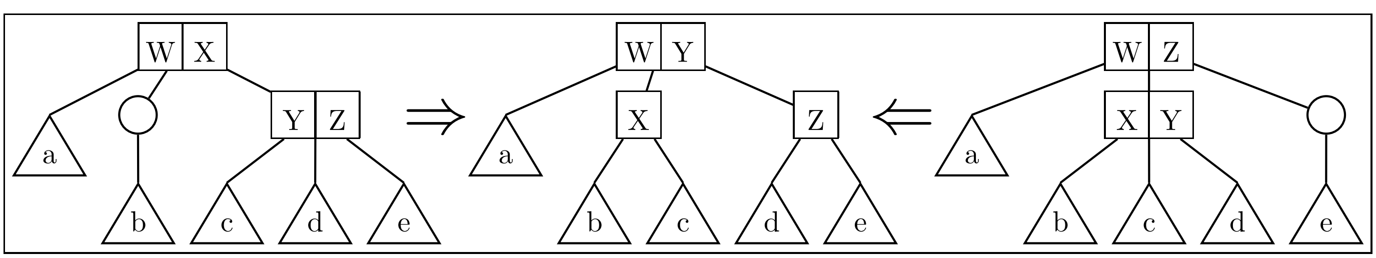


4. *The hole has a 3-node as a parent and a 3-node as a sibling.* Again there are two subcases.

   (a) The first subcase involves subtrees a, b, c, and d, whose heights are one less than that of subtree e.

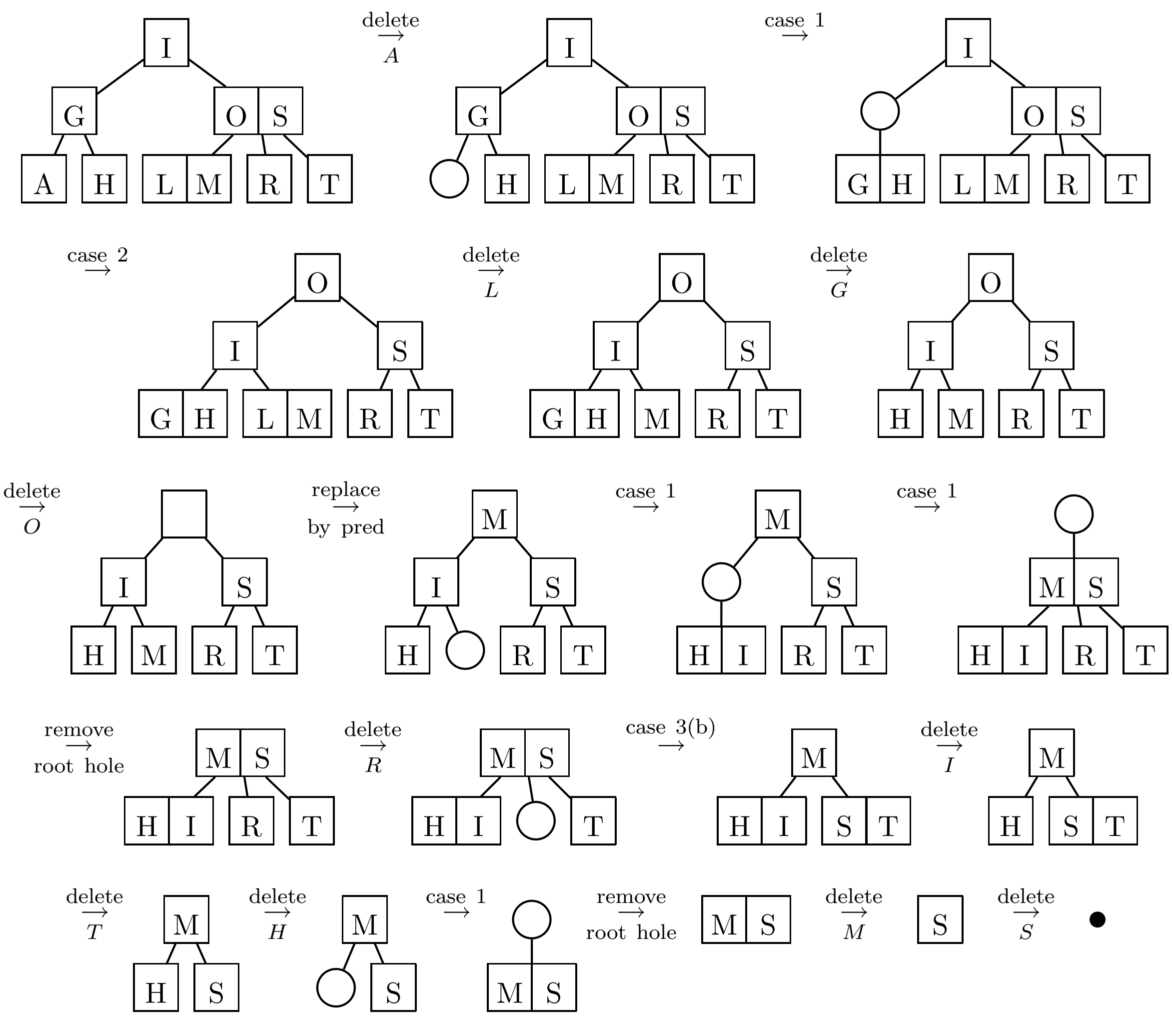


   (b) The second subcase involves subtrees b, c, d, and e, whose heights are one less than that of subtree a. When the hole is in the middle, there may be ambiguity in terms of whether to apply the right-hand rule of the first subcase or the left-hand rule of the second subcase. Either application is OK.

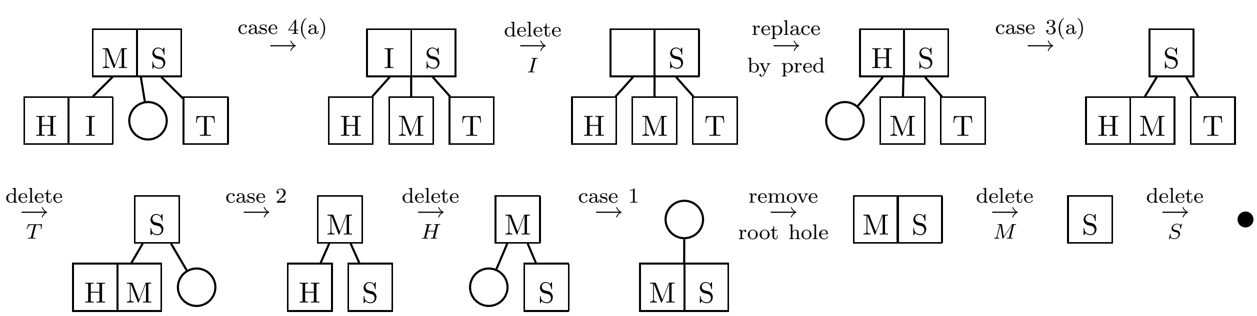


2-3 Tree Deletion Example

Here we show the letter-by-letter deletion of the letters `A L G O R I T H M S` from the final 2-3 tree of the insertion example:



At the point where case 3(b) was applied, it is also possible to apply case 4(a) instead. This leads to the following alternative deletion sequence:



There are some other choices in the above sequences. In each of the spots where a deleted value in a non-terminal node was replaced by its in-order predecessor, it could have been replaced by its in-order successor.