**CS231 Algorithms**
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**Handout #7**
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# COUNTING AND PROBABILITY

**Reading:** CLRS Chapter 5 & Appendix C; CLR Sections 6.1 -- 6.3; 6.6.

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## Counting

Given a set S with n elements, how many ways are there to choose k elements from S? The answer depends on whether (1) order matters and (2) duplicates are allowed.

| | **Duplicates not allowed** **($k \leq n$)** | **Duplicates allowed** **(k may be > n)** |
|---|---|---|
| **Order matters (sequences)** | *k-permutation* $\frac{n!}{(n - k)!}$ | *k-tuple (k-string)* $n^k$ |
| **Order doesn't matter (sets)** | *k-combination* $\frac{n!}{k!\,(n - k)!}$ | *k-selection* $\frac{(n - 1 + k)!}{k!\,(n - 1)!}$ |

Example: S = {a, b , c, d}, k = 2

| Type | Elements | Number |
|---|---|---|
| 2-permutations of S | | |
| 2-combination of S | | |
| 2-strings of S | | |
| 2-selections of S | | |

Exercise: Try k = 3

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**Sample Spaces**

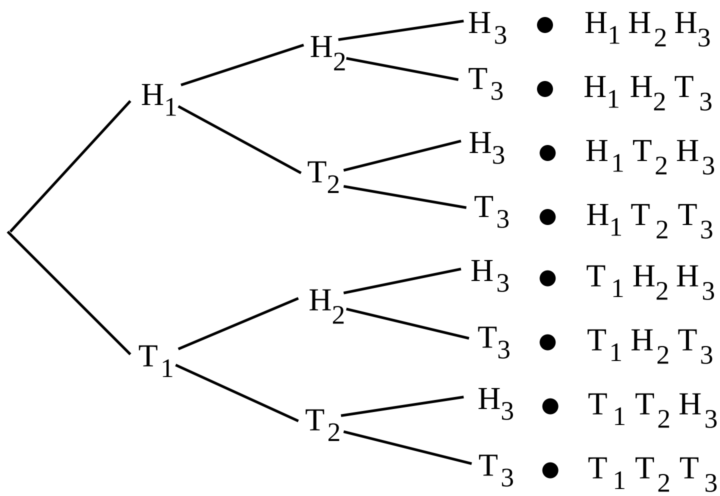
A **sample space** is the set of all possible outcomes of an experiment.

Experiment 1: Flip three coins.
Sample space =

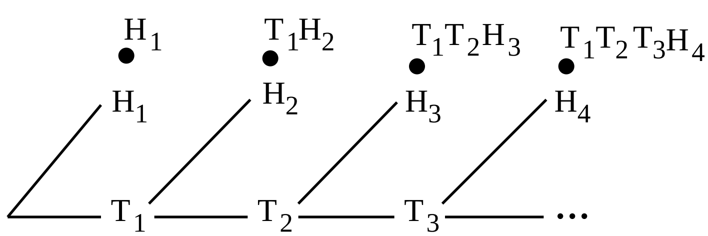
Experiment 2: Flip a coin until you get heads.
Sample space =

A sequential sample spaces describing a sequence of experiments is naturally viewed as a tree:

Experiment 1:




Experiment 2:

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**Events**

An **event** is a subset of a sample space.

Experiment 1:

Event 1a: First flip is a head =

Event 1b: Second flip is a tail =

Event 1c: Exactly two tails =

Event 1d: Two consecutive flips the same =

Experiment 2:

Event 2a: First flip is a head =

Event 2b: First flip is a tail =

Event 2c: Even number of flips =

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**Event Combinators**

If A and B are events in sample space S,

"A and B" is translated "A ∩ B"

"A or B" is translated "A     B"

"not A" is translated "S − A"

Two events A and B are **mutually exclusive** if A ∩ B =    .

Of the four events in Experiment 1, which pairs are mutually exclusive?

Of the three events in Experiment 2, which pairs are mutually exclusive?

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**Probability Distributions**

A **probability distribution** Pr{} on a sample space S is any mapping from events to [0..1] such that the following axioms hold:

1. Pr{A} ≥ 0 for any event A.
2. Pr{S} = 1
3. If A ∩ B =   then Pr{A   B} = Pr{A} + Pr{B}

Some useful theorems:

Pr{  } = 0.

Pr{S – A} = 1 – Pr{A}.

If A   B then Pr{A} ≤ Pr{B}

Pr{A   B} = Pr{A} + Pr{B} – Pr{A ∩ B}

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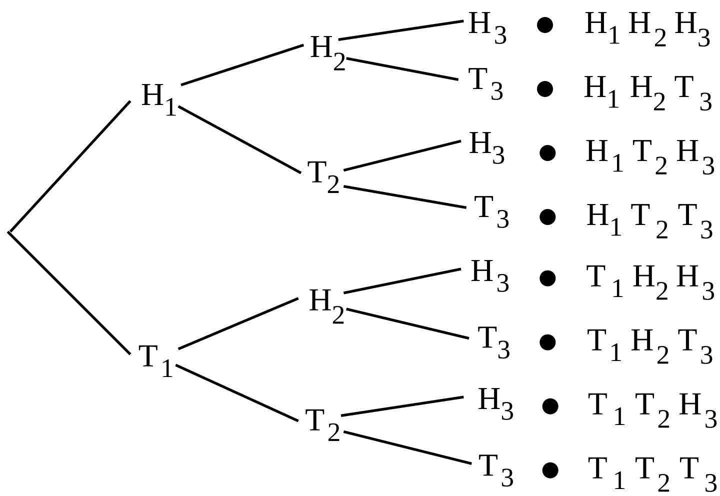
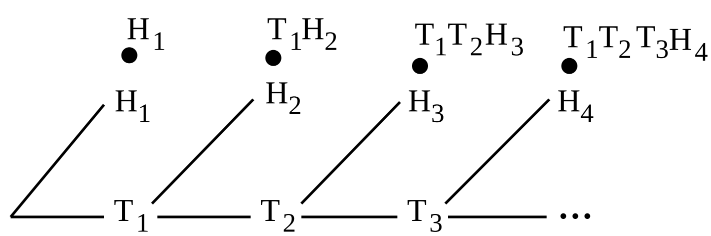
**Probability Trees**

A sequential sample space gives rise to a **probability tree**. If events at distinct stages of tree are independent (defined below), probability of leaf is product of probabilities on path to leaf.

Experiment 1 with $H_1 = (1/3)$, $H_2 = (1/4)$, $H_3 = (1/5)$




Experiment 2 with $H_i = (1/3)$




The probability of an event in a sequential sample space is the sum of the probability of its member leaves in the probability tree. What are the probabilities of the following events?

1a:

1b:

1c:

1d:

2a:

2b:

2c:

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**Birthday Pardox**

How many people must be in a room before there is a 50% chance that two of them have the same birthday?  A 90% chance?   A 99% chance? (Assume that for any given person, each day of the year is equally likely as a birthday.  Ignore leap years.)

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**Conditional Probability**

Probability of A given B = $\Pr\{A \mid B\} = \dfrac{\Pr\{A \cap B\}}{\Pr\{B\}}$

*Example 1*: In Experiment 1 with $H_1 = (1/3)$, $H_2 = (1/4)$, $H_3 = (1/5)$, what is probability of 1c (exactly two tails) given 1d (two consecutive flips the same)?

*Example 2:* Ann has two children, one of which is a girl. What is the probability that the other child is a boy? (Assume a child is girl or boy with equal probability.)

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**Independence**

Events A and B are **independent** if $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$.
(Equivalently, $\Pr\{A \mid B\} = \Pr\{A\}$)

Of the four events in Experiment 1, which pairs are independent?

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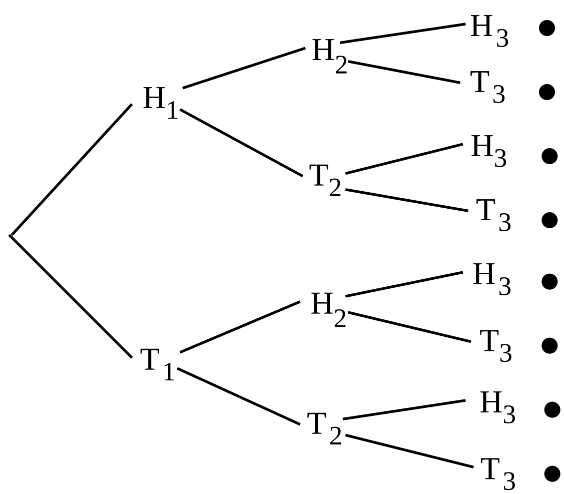
## Discrete Random Variables

A discrete random variable on a sample space S is a function f: S $\rightarrow$ Real.

E.g., in Experiment 1 let

$h(x)$ = number of heads in outcome x.

$r(x)$ = length of longest run in outcome x.




| Outcomes | h | r |
|---|---|---|
| $H_1 H_2 H_3$ | | |
| $H_1 H_2 T_3$ | | |
| $H_1 T_2 H_3$ | | |
| $H_1 T_2 T_3$ | | |
| $T_1 H_2 H_3$ | | |
| $T_1 H_2 T_3$ | | |
| $T_1 T_2 H_3$ | | |
| $T_1 T_2 T_3$ | | |

In Experiment 2, let $f(x)$ = the number of flips for outcome x.

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**Probability Density Function**

The **pre-image** of a value v under f = $f^{-1}(v) = \{x \quad S \mid f(x) = v\}$
(This differs from CLR's notation "f = v", which I find confusing.)

Each pre-image is an event:

$h^{-1}(1) =$

$r^{-1}(3) =$

$f^{-1}(2) =$

The **probability density function (PDF)** of discrete random variable f is another function that maps each target value v of f to its probability. This is determined by summing the probabilities of all sources x that f maps to v:

$$[PDF(f)](v) = Pr\{f^{-1}(v)\} = \sum_{\{x \quad S \mid f(x) = v\}} Pr\{x\}$$

What is the function PDF(h)?

What is the function PDF(r)?

What is the function PDF(f)?

Discrete random variables f and g are **independent** if for all w and v, the events $f^{-1}(w)$ and $g^{-1}(v)$ are independent.

Are h and r independent?

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**Expected Value**

The **expected value** E[f] of a discrete random variable f is a weighted average in which target values v of f are weighted by their probability:

$$E[f] = \sum_{v} v \ \Pr\{f^{-1}(v)\}$$

Examples:

Expected number of heads in Experiment 1 = E[h] =

Suppose that you "play" Experiment 1 as a game in which you get $1 for every head. How much would you be willing to pay to play such a game?

Expected length of runs in Experiment 1 = E[r] =

Expected number of flips in Experiment 2 = E[f] =

Expectations are **linear**. That is, if f & g are discrete random variables and c is a constant:

E[f + g] = E[f] + E[g] {The sum of functions f and g is a function (f + g)(x) = f(x) + g(x).}

E[cf] = c[E[f]] {The scaling by c of a function f is a function (cf)(x) = cf(x).}

If f and g are independent, then

E[fg] = E[f]E[g] {The product of functions f and g is a function (fg)(x) = f(x)g(x).}

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