Wellesley College ◊ CS231 Algorithms ◊ September 6, 1996
Handout #5

ASYMPTOTICS AND FUNCTIONS

Note: This handout summarizes highlights of CLR Chapter 2. See the book for more details.

Motivation

Fine-grained bean counting exposes too much detail for comparing functions.

Want a course-grained way to compare functions that ignores constant factors and focuses on their relative growth in the limit as input sizes get large.

For example, consider:

	n = 1	n = 1,000	n = 1,000,000
$p(n) = 100n + 1000$			
$q(n) = 3n^2 + 2n + 1$			
$r(n) = 0.1n^2$			

How Do Your Functions Grow?

Asymptotic notation is a way of characterizing functions that facilitates comparing their growth in the limit of large inputs. Here is an informal summary of the notation:

Notation	Pronunciation	Loosely
f ω(g)	f is way bigger than g	f > g
f Ω(g)	f is at least as big as g	f ≥ g
f Θ(g)	f is about the same as g	f = g
f O(g)	f is at most as big as g	f ≤ g
f o(g)	f is way smaller than g	f < g

Notes:

- Each of ω(g), Ω(g), Θ(g), O(g), o(g) denotes a *set* of functions. Thus, ω(g) is the set of all functions way bigger than g, Ω(g) is the set of all functions at least as big as g, etc.

- The notation f = ω(g) is really shorthand for f ω(g).

- The phrases "is at least O(...)" and "is at most Ω(...)" are non-sensical. "Is at least" should be written Ω, and "is at most" should be written O.

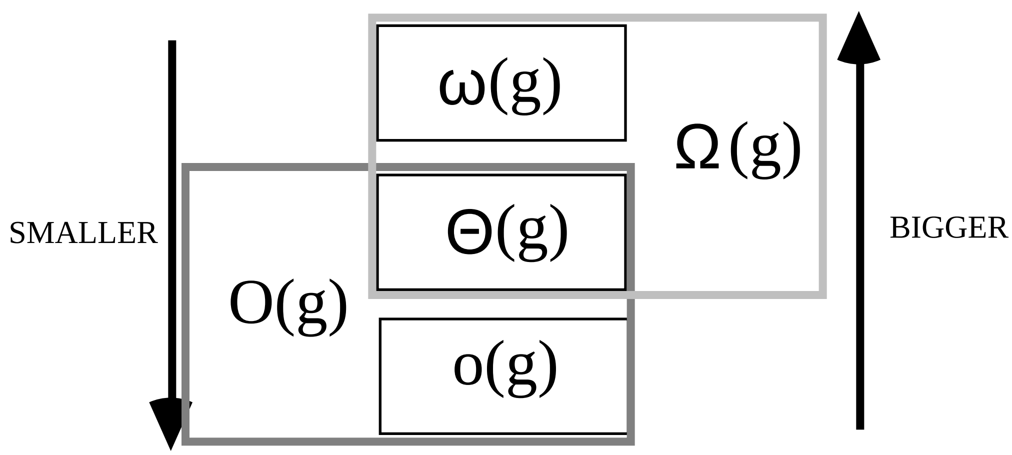
Relating the Notations

Here are some of the relationships between the notations:

> If f ω(g), then f Ω(g).
> If f o(g), then f O(g).
> Ω(g) ω(g) Θ(g)
> O(g) o(g) Θ(g)
> Θ(g) = Ω(g) ∩ O(g)
> f ω(g) if and only if g o(f)
> f Ω(g) if and only if g O(f)
> f Θ(g) if and only if g Θ(f)

Warning: unlike numbers, not every pair of functions is comparable!

The following diagram depicts some of these relationships:



Formalizing the Non-tight Bounds (o and ω)

f $o(g)$ if $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$

f $\omega(g)$ if $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$

Formalizing the Tight Bounds (O, Ω, and Θ)

$O(g) = \{f \mid$ there exist positive constants c, n_0 such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0$.$\}$

Think of this as a game. Suppose you claim that f $O(g)$. Then you select c and n_0 , but I try to find a particular n that defeats your claim.

$\Omega(g) = \{f \mid$ there exist positive constants c, n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0$.$\}$

$\Theta(g) = \{f \mid$ there exist positive constants c_1 , c_2 , n_0 such that
$0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0$.$\}$

Exponentials

Notation:

- a^n = the product of *n* copies of *a*.
- $a^{-n} = \frac{1}{a^n}$

Key Identities:

- $a^m a^n = a^{m+n}$ {Special case: $a^0 = 1$.}
- $(a^m)^n = a^{mn} = (a^n)^m$

Examples:

$(5^2)^3 =$

$5^2 \ 5^3 =$

$5^2 \ +5^3 =$

$25^{(\frac{3}{2})} =$

Logarithms

Notation:

- $\log_b a$ = the power to which b must be raised to equal a. (More loosely, it is the number of times that a can be divided by b to reach 1.)
- log**Error!**) = −log**Error!** a
- $\lg n = \log_2 n$
- $\ln n = \log_e n$
- $\lg^k(n) = (\lg n)^k$

Key Identities (duals of exponential identities):

- $\log_c(ab) = \log_c(a) + \log_c(b)$ {Special case: $\log_c 1 = 0$}

  {Special case: $\log_c a^n = n \ \log_c a$}
- $\log_c(a) = \log_c(b) \ \log_b(a)$

Examples:

$\lg(2n^3) =$

$\ln(32) =$

Relating Exponentials and Logarithms

Key Identities:

- $b^{(\log_b a)} = a = \log_b(b^a)$

Examples:

lg **Error!**) =

$32^{(\lg n)}$ =

Asymptotics Involving Exponentials and Logarithms

How do $\log_2 n$ and $\log_3 n$ compare?

How do 2^n and 3^n compare?

Fact 1: if a > 0, $\lim_{n \to \infty} \left(\frac{a^n}{n^b} \right) = \infty$

Fact 1 implies $a^n \quad \omega(n^b)$.
In other words: *Any positive exponential grows faster than any polynomial.*

Substituting lg n for n and 2^a for a in Fact 1 yields:

Fact 2: if a > 0, $\lim_{n \to \infty} \left(\frac{n^a}{\lg^b n} \right) = \infty$

Fact 2 implies $n^a \quad \omega(\lg^b n)$.
In other words: *Any positive polynomial grows faster than any polylogarithmic function.*

Factorials

Definition: n! = 1 2 3 … n

Stirling's approximation: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$

Asymptotics derivable from Stirling's approximation:

- $n! = o(n^n)$

- $n! = \omega(2^n)$
- $\lg(n!) = \Theta(n \lg n)$
