

COUNTING AND PROBABILITY

Reading: CLR Sections 6.1 -- 6.3; 6.6.

Counting

Given a set S with n elements, how many ways are there to choose k elements from S? The answer depends on whether (1) order matters and (2) duplicates are allowed.

	Duplicates not allowed (k ≤ n)	Duplicates allowed (k may be > n)
Order matters (sequences)	<i>k-permutation</i> $\frac{n!}{(n - k)!}$	<i>k-tuple (k-string)</i> n^k
Order doesn't matter (sets)	<i>k-combination</i> $\frac{n!}{k! (n - k)!}$	<i>k-selection</i> $\frac{(n - 1 + k)!}{k! (n - 1)!}$

Example: S = {a, b, c, d}, k = 2

Type	Elements	Number
2-permutations of S		
2-combination of S		
2-strings of S		
2-selections of S		

Exercise: Try k = 3

Events

An **event** is a subset of a sample space.

Experiment A:

Event A1: First flip is a head =

Event A2: Second flip is a tail =

Event A3: Exactly two tails =

Event A4: Two consecutive flips the same =

Experiment B:

Event B1: First flip is a head =

Event B2: First flip is a tail =

Event B3: Even number of flips =

If A and B are events in sample space S,

"A and B" is translated " $A \cap B$ "

"A or B" is translated " $A \cup B$ "

"not A" is translated " $S - A$ "

Two events A and B are **mutually exclusive** if $A \cap B = \emptyset$.

Of the four events in Experiment A, which pairs are mutually exclusive?

Of the three events in Experiment B, which pairs are mutually exclusive?

Probability Distributions

A **probability distribution** $\Pr\{\cdot\}$ on a sample space S is any mapping from events to $[0..1]$ such that the following axioms hold:

1. $\Pr\{A\} \geq 0$ for any event A .
2. $\Pr\{S\} = 1$
3. If $A \cap B = \emptyset$ then $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$

Some useful theorems:

$$\Pr\{\emptyset\} = 0.$$

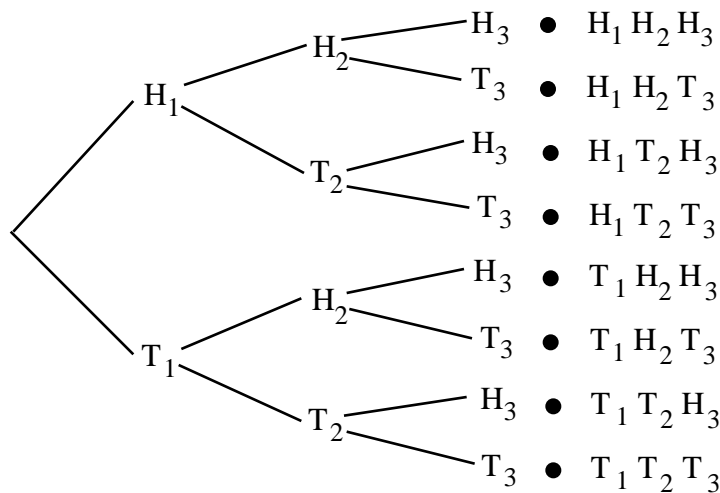
$$\Pr\{S - A\} = 1 - \Pr\{A\}.$$

$$\text{If } A \cap B = \emptyset \text{ then } \Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$$

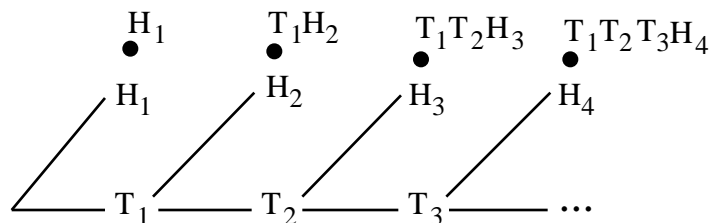
$$\Pr\{A \cap B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cup B\}$$

A sequential sample space gives rise to a **probability tree**. If events at distinct stages of tree are independent (defined below), probability of leaf is product of probabilities on path to leaf.

Experiment A with $H_1 = (1/3)$, $H_2 = (1/4)$, $H_3 = (1/5)$



Experiment B with $H_i = (1/3)$



Conditional Probability

$$\text{Probability of A given B} = \Pr\{A | B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

Example 1: In Experiment A with $H_1 = (1/3)$, $H_2 = (1/4)$, $H_3 = (1/5)$, what is probability of A3 (exactly two tails) given A4 (two consecutive flips the same)?

Example 2: Ann has two children, one of which is a girl. What is the probability that the other child is a boy? (Assume a child is girl or boy with equal probability.)

Independence

Events A and B are **independent** if $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$.
(Equivalently, $\Pr\{A | B\} = \Pr\{A\}$)

Of the four events in Experiment A, which pairs are independent?

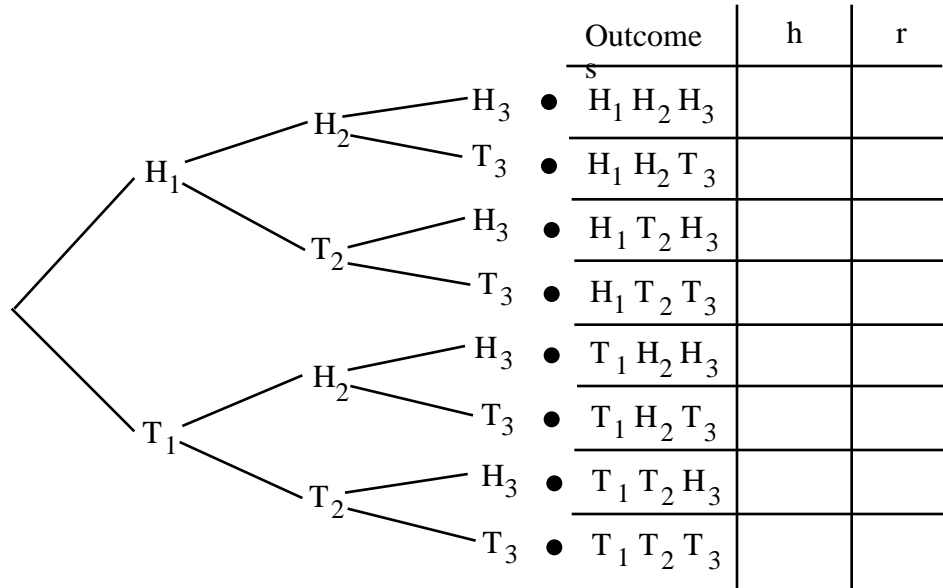
Discrete Random Variables

A discrete random variable on a sample space S is a function $f: S \rightarrow \text{reals}$.

E.g., in Experiment A let

$h(x)$ = number of heads in outcome x .

$r(x)$ = length of longest run in outcome x .



Probability Density Function

The **pre-image** of a value v under $f = f^{-1}(v) = \{x \in S \mid f(x) = v\}$
(This differs from CLR's notation " $f = v$ ", which I find confusing.)

Each pre-image is an event:

$$h^{-1}(1) =$$

$$r^{-1}(3) =$$

The **probability density function (PDF)** of discrete random variable f is another function that maps each target value v of f to its probability. This is determined by summing the probabilities of all sources x that f maps to v :

$$[\text{PDF}(f)](v) = \Pr\{f^{-1}(v)\} = \sum_{\{x \in S \mid f(x) = v\}} \Pr\{x\}$$

What is the function $\text{PDF}(h)$?

What is the function $\text{PDF}(r)$?

Discrete random variables f and g are **independent** if for all w and v , the events $f^{-1}(w)$ and $g^{-1}(v)$ are independent.

Are h and r independent?

Expected Value

The **expected value** $E[f]$ of a discrete random variable f is a weighted average in which target values v of f are weighted by their probability:

$$E[f] = \sum_v v \Pr\{f^{-1}(v)\}$$

Examples:

Expected number of heads in Experiment A = $E[h] =$

Expected length of runs in Experiment A = $E[r] =$

Expected number of flips in Experiment B =

Expectations are linear. If f & g are discrete random variables and c is a constant:

$$E[f + g] = E[f] + E[g] \quad \{\text{The sum of functions } f \text{ and } g \text{ is a function } (f + g)(x) = f(x) + g(x).\}$$

$$E[cf] = c[E[f]] \quad \{\text{The scaling by } c \text{ of a function } f \text{ is a function } (cf)(x) = cf(x).\}$$

If f and g are independent, then

$$E[fg] = E[f]E[g] \quad \{\text{The product of functions } f \text{ and } g \text{ is a function } (fg)(x) = f(x)g(x).\}$$
