Topological Sort

Digraphs

- **Directed graphs, digraphs**, are graphs in which edges connecting nodes are one-way.

- Directed graphs may be represented using either an adjacency matrix or an adjacency list.
## Adjacency-Matrix of $G(V,E)$

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## Adjacency List of $G(V,E)$

### From

- A: B, F, H, I
- B: A, C, D
- C: B, G
- D: B, E
- E: D, J
- F: A, G, H, I
- G: F, C, H, I
- H: F, G, I, J
- I: F, G, H
- J: E, H
- K: G, L

### To

- A: B, F, H, I
- B: A, C, D
- C: B, G
- D: B, E
- E: D, J
- F: A, G, H, I
- G: F, C, H, I
- H: F, G, I, J
- I: F, G, H
- J: E, H
- K: G, L
- L: K, M
- M: K

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From and To matrices and adjacency lists for the graph $G(V,E)$.
Setting Up Depth-First Search

DFS(G)

for each u ∈ G.V
    u.color = WHITE
    u.parent = NIL

    time = 0

    for each u ∈ G.V
        if u.color == WHITE
            DFS-Visit(G,u)

DFS-Visit(u)

DFS-Visit(G,u)

time = time + 1    // u has just been discovered
u.color = GRAY
u.d = time

    for each v ∈ G.Adj[u]    // explore (u,v)
        if v.color == WHITE
            v.parent = u
            DFS-Visit(G,v)

u.color = BLACK    // finished u

time = time + 1
u.f = time
Depth-First Search Forest*

*Which edges betray a directed cycle?

Professor Bumstead Gets Dressed*

*The good Professor wishes to order the vertices so that no vertex comes before any vertex that points to it.
Topological Sort

Topological-Sort(G) 
call DFS(G) to compute finishing times f[v] 
for each vertex v 
as each vertex is finished, insert it onto the 
front of the linked list 
return the linked list of vertices.

All Dressed Up*

*In order for the Professor’s algorithm to work, what must be true about the digraph?
Proof of Correctness

**Theorem**
Topological-Sort($G$) produces a topological sort of a directed acyclic graph.

Recognizing DAGs

**Lemma**
A directed graph $G$ is acyclic if and only if a depth-first search of $G$ yields no back edges.