Lecture 3 – Asymptotic Notation
Reading: KT Sections 2.1 and 2.2

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Agenda

2. Algorithm Analysis
   - computational tractability
   - asymptotic order of growth
   - survey of common running times
Algorithm efficiency

• What makes us say that an algorithm is efficient?
  • Real answer: when it’s better than its brute force counter-part

  Brute force. For many nontrivial problems, there is a natural brute-force search algorithm that checks every possible solution.
  • Typically takes $2^n$ time or worse for inputs of size $n$.
  • Unacceptable in practice.

Polynomial time algorithms

We say that an algorithm is efficient if it has a polynomial running time.

Justification. It really works in practice!
• In practice, the poly-time algorithms that people develop have low constants and low exponents.
• Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions. Some poly-time algorithms do have high constants and/or exponents, and/or are useless in practice.

Q. Which would you prefer $20n^{20}$ vs. $n^{1 + 0.02}n$?
Worst case analysis

Worst case. Running time guarantee for any input of size $n$.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Exceptions. Some exponential-time algorithms are used widely in practice because the worst-case instances seem to be rare.

Other types of analyses

Worst case. Running time guarantee for any input of size $n$.
Ex. Heapsort requires at most $2n \log_2 n$ compares to sort $n$ elements.

Probabilistic. Expected running time of a randomized algorithm.
Ex. The expected number of compares to quicksort $n$ elements is $\sim 2n \ln n$.

Amortized. Worst-case running time for any sequence of $n$ operations.
Ex. Starting from an empty stack, any sequence of $n$ push and pop operations takes $O(n)$ primitive computational steps using a resizing array.

Average-case. Expected running time for a random input of size $n$.
Ex. The expected number of character compares performed by 3-way radix quicksort on $n$ uniformly random strings is $\sim 2n \ln n$.

Also. Smoothed analysis, competitive analysis, ...
The way things grow

![Graph showing cost vs. size with linear, logarithmic, constant, quadratic, and exponential functions]

By the numbers

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>4 sec</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>18 min</td>
<td>10$^{15}$ years</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>10$^{17}$ years</td>
<td>very long</td>
<td></td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td></td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>&lt; 1 sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td></td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td></td>
</tr>
</tbody>
</table>
2. Algorithm Analysis

- computational tractability
- asymptotic order of growth
- survey of common running times

Agenda

Big-Oh notation

**Upper bounds.** \( T(n) \) is \( O(f(n)) \) if there exist constants \( c > 0 \) and \( n_0 \geq 0 \) such that \( T(n) \leq c \cdot f(n) \) for all \( n \geq n_0 \).

- This means that \( T(n) \) grows no faster than \( f(n) \).
- For example, let’s consider \( 17n^2 \) and \( n^2 \)

Can you find \( c \) and \( n_0 \)?
Common mistakes

Equals sign. $O(f(n))$ is a set of functions, but computer scientists often write $T(n) = O(f(n))$ instead of $T(n) \in O(f(n))$.

Ex. Consider $f(n) = 5n^3$ and $g(n) = 3n^2$.
- We have $f(n) = O(n^3) = g(n)$.
- Thus, $f(n) = g(n)$. ×

Domain. The domain of $f(n)$ is typically the natural numbers $\{0, 1, 2, \ldots\}$.
- Sometimes we restrict to a subset of the natural numbers.
- Other times we extend to the reals.

Non-negative functions. When using big-Oh notation, we assume that the functions involved are (asymptotically) non-negative.

Big-Omega notation

Lower bounds. $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $T(n) \geq c \cdot f(n)$ for all $n \geq n_0$.

Typical usage. Any compare-based sorting algorithm requires $\Omega(n \log n)$ compares in the worst case.

Meaningless statement. Any compare-based sorting algorithm requires at least $O(n \log n)$ compares in the worst case.
Big-Theta notation

**Tight bounds.** $T(n)$ is $\Theta(f(n))$ if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$ for all $n \geq n_0$.

**Ex.** $T(n) = 32n^2 + 17n + 1$.
- $T(n)$ is $\Theta(n^2)$, \[\text{choose } c_1 = 32, c_2 = 50, n_0 = 1\]
- $T(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$.

**Typical usage.** Mergesort makes $\Theta(n \log n)$ compares to sort $n$ elements.

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Some properties to know...

**Polynomials.** Let $T(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then, $T(n)$ is $\Theta(n^d)$.

**Pf.** \[\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0\]

**Logarithms.** $\Theta(\log_a n)$ is $\Theta(\log_b n)$ for any constants $a, b > 0$, \[\text{no need to specify base assuming it is a constant}\]

**Logarithms and polynomials.** For every $d > 0$, $\log n$ is $O(n^d)$.

**Exponentials and polynomials.** For every $r > 1$ and every $d > 0$, $n^d$ is $O(r^n)$.

**Pf.** \[\lim_{n \to \infty} \frac{n^d}{r^n} = 0\]
Can you think of a?

- Linear time algorithm
- Sublinear time algorithm
- Linearithmic time algorithm
- Quadratic time algorithm
- Cubic time algorithm