Lecture 4 – The Stable Matching Problem
Reading: KT Sections 1.2 and 2.3

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Matching Residents to Hospitals

- **Goal.** Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

- **Unstable pair:** applicant \( x \) and hospital \( y \) are unstable if:
  - \( x \) prefers \( y \) to its assigned hospital.
  - \( y \) prefers \( x \) to one of its admitted students.

- **Stable assignment.** Assignment with no unstable pairs.
  - Natural and desirable condition.
  - Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

- **Input:**
  - Given \( n \) residents and \( n \) hospitals, with their rating of each other.
  - Each resident lists hospitals in order of preference from best to worst.
  - Each hospital lists residents in order of preference from best to worst.

- **Goal:**
  - Find a "suitable" matching.

<table>
<thead>
<tr>
<th>Residents' Preference Profile</th>
<th>Hospitals' Preference Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>favorite</strong></td>
<td><strong>least favorite</strong></td>
</tr>
<tr>
<td>Xavier</td>
<td>Amy</td>
</tr>
<tr>
<td>Yancey</td>
<td>Bertha</td>
</tr>
<tr>
<td>Zeus</td>
<td>Amy</td>
</tr>
</tbody>
</table>

Stable Matching Problem

- **Perfect matching:** everyone is matched monogamously.
  - Each resident gets exactly one hospital.
  - Each hospital gets exactly one resident.

- **Stability:** no incentive for some pair of participants to undermine assignment by joint action.
  - In matching \( M \), an unmatched pair \( m-w \) is unstable if ...
  - Unstable pair \( r-h \) could each improve by breaking contracts.

- **Stable matching:** perfect matching with no unstable pairs.

- **Stable matching problem.** Given the preference lists of \( n \) residents and \( n \) hospitals, find a stable matching if one exists.
Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?


- An unstable pair (r,h) could each improve by joint action.

Stable Matching Problem

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Stable Roommate Problem

- Q. Do stable matchings always exist?

- A. Not obvious a priori.

- Stable roommate problem.
  - 2n people; each person ranks others from 1 to 2n-1.
  - Assign roommate pairs so that no unstable pairs.

- Observation. Stable matchings do not always exist for stable roommate problem.
Propose-And-Reject Algorithm

- The (Gale-Shapley 1962) deferred acceptance algorithm is an intuitive method that guarantees to find a stable matching.

Initially all \( m \in M \) and \( w \in W \) are free
While there is a man \( m \) who is free and hasn’t proposed to every woman
Choose such a man \( m \)
Let \( w \) be the highest-ranked woman in \( m \)'s preference list
to whom he has not yet proposed
If \( w \) is free then
\( (m, w) \) become engaged
Else \( w \) is currently engaged to \( m' \)
If \( w \) prefers \( m \) to \( m' \)
\( m \) remains free
Else \( w \) prefers \( m \) to \( m' \)
\( (m, w) \) become engaged
\( m' \) becomes free
Endif
Endif
Endwhile
Return the set \( S \) of engaged pairs

Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.
- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."
- Claim. Algorithm terminates after at most \( n^2 \) iterations of while loop.
- Pf. Each time through the while loop a man proposes to a new woman. There are only \( n^2 \) possible proposals.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Man} & A & B & C & D & E \\
\hline
\text{Woman} & A & B & C & D & E \\
\hline
\end{array}
\]

\( n-1 \times n \) proposals required

Proof of Correctness: Perfection

- Claim. All men and women get matched.
- Pf. (by contradiction)
  - Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2, Amy was never proposed to.
  - But, Zeus proposes to everyone, since he ends up unmatched.

Proof of Correctness: Stability

• Claim. No unstable pairs in a matching $S^*$.  
• Pf. (by contradiction) 
  • Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching $S^*$. 
  • Case 1: Z never proposed to A. 
    $$\Rightarrow$$ Z prefers his GS partner to A. 
    $$\Rightarrow$$ A-Z is stable. 
  • Case 2: Z proposed to A. 
    $$\Rightarrow$$ A rejected Z (right away or later) 
    $$\Rightarrow$$ A prefers her GS partner to Z. 
    $$\Rightarrow$$ A-Z is stable. 

  • In either case A-Z is stable, a contradiction. •

Summary

• Stable matching problem.  
  • Given n men and n women, and their preferences, find a stable matching if one exists. 

  • Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance. 

  • Q. How to implement GS algorithm efficiently? 

  • Q. If there are multiple stable matchings, which one does GS find?