Reminders

- Assignment 1 due tonight
  - Use Gradescope

- Assignment 2 will be posted tonight

- Please go to discussion section on Sunday at 6pm in E211
  - You will trace all algorithms discussed in class, and work on more examples

Stable matching problem

Our first algorithm 😊
Matching Residents to Hospitals

- **Goal.** Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

- **Unstable pair:** applicant $x$ and hospital $y$ are unstable if:
  - $x$ prefers $y$ to its assigned hospital.
  - $y$ prefers $x$ to one of its admitted residents.

- **Stable assignment.** Assignment with no unstable pairs.
  - Natural and desirable condition.
  - Individual self-interest will prevent any applicant/hospital deal from being made.

---

Stable Matching Problem

- **Input:**
  - Given $n$ residents and $n$ hospitals, with their rating of each other.
  - Each resident lists hospitals in order of preference from best to worst.
  - Each hospital lists residents in order of preference from best to worst.

- **Goal:**
  - Find a "suitable" matching.

<table>
<thead>
<tr>
<th>Residents’ Preference Profile</th>
<th>Hospitals’ Preference Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1st</strong></td>
<td><strong>2nd</strong></td>
</tr>
<tr>
<td>Xavier</td>
<td>Amy</td>
</tr>
</tbody>
</table>
Some definitions ...

- **Perfect matching**: everyone is matched monogamously.
  - Each resident gets exactly one hospital.
  - Each hospital gets exactly one resident.

- **Stability**: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching M, an unmatched pair r-h is unstable if ...
  - Unstable pair r-h could each improve by breaking contracts.

- **Stable matching**: perfect matching with no unstable pairs.

- **Stable matching problem**: Given the preference lists of n residents and n hospitals, find a stable matching if one exists.

Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?
Stable Matching Problem

• Q. Is assignment X-C, Y-B, Z-A stable?
• A. No. Bertha and Xavier would break contract.

• An unstable pair \((r,h)\) could each improve by joint action.

<table>
<thead>
<tr>
<th>favorite</th>
<th>least favorite</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2nd</td>
</tr>
<tr>
<td>Xavier</td>
<td>Amy</td>
</tr>
<tr>
<td>Yancey</td>
<td>Bertha</td>
</tr>
<tr>
<td>Zeus</td>
<td>Amy</td>
</tr>
</tbody>
</table>

Residents’ Preference Profile

<table>
<thead>
<tr>
<th>favorite</th>
<th>least favorite</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2nd</td>
</tr>
<tr>
<td>Amy</td>
<td>Yancey</td>
</tr>
<tr>
<td>Bertha</td>
<td>Xavier</td>
</tr>
<tr>
<td>Clare</td>
<td>Xavier</td>
</tr>
</tbody>
</table>

Hospitals’ Preference Profile

Stable Matching Problem

• Q. Is assignment X-A, Y-B, Z-C stable?
• A. Yes.

<table>
<thead>
<tr>
<th>favorite</th>
<th>least favorite</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2nd</td>
</tr>
<tr>
<td>Xavier</td>
<td>Amy</td>
</tr>
<tr>
<td>Yancey</td>
<td>Bertha</td>
</tr>
<tr>
<td>Zeus</td>
<td>Amy</td>
</tr>
</tbody>
</table>

Residents’ Preference Profile

<table>
<thead>
<tr>
<th>favorite</th>
<th>least favorite</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2nd</td>
</tr>
<tr>
<td>Amy</td>
<td>Yancey</td>
</tr>
<tr>
<td>Bertha</td>
<td>Xavier</td>
</tr>
<tr>
<td>Clare</td>
<td>Xavier</td>
</tr>
</tbody>
</table>

Hospitals’ Preference Profile
Stable Roommate Problem

• Q. Do stable matchings always exist?
  • A. Not obvious a priori.

• Stable roommate problem.
  • 2n people; each person ranks others from 1 to 2n-1.
  • Assign roommate pairs so that no unstable pairs.

<table>
<thead>
<tr>
<th></th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>Bob</td>
<td>C</td>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>Chris</td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>Doofus</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

• Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

• The [Gale-Shapley 1962] deferred acceptance algorithm is an intuitive method that guarantees to find a stable matching.

Initially all $m \in M$ and $w \in W$ are free
While there is a man $m$ who is free and hasn’t proposed to every woman
Choose such a man $m$
Let $w$ be the highest-ranked woman in $m$’s preference list
to whom $m$ has not yet proposed
If $w$ is free then
$(m, w)$ become engaged
Else $w$ is currently engaged to $m’$
  If $w$ prefers $m’$ to $m$ then
    $m$ remains free
  Else $w$ prefers $m$ to $m’$
    $(m, w)$ become engaged
    $m’$ becomes free
Endif
Endif
Endwhile
Return the set $S$ of engaged pairs
So far...

• **Stable matching problem.**
  • Given $n$ men and $n$ women, and their preferences, find a stable matching if one exists.

• Gale-Shapley algorithm.

• Q. Does it correctly find a stable matching for any problem instance?

• Q. How to implement GS algorithm efficiently?

• Q. If there are multiple stable matchings, which one does GS find?
So far ...

- Stable matching problem.
  - Given n men and n women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm.

Q. How to implement GS algorithm efficiently?

Q. Does it correctly find a stable matching for any problem instance?

Q. If there are multiple stable matchings, which one does GS find?

Let’s discuss implementation
Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.

- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

- Claim. Algorithm terminates after at most $n^2$ iterations of while loop.

- Pf. Each time through the while loop a man proposes to a new woman. There are only $n^2$ possible proposals.

\[ n(n^2) + 1 \text{ proposals required} \]

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- An instance with two stable matchings.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.
Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

- Man-optimal assignment. Each man receives best valid partner.

- Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- Claim. All executions of GS yield woman-pessimal assignment, which is a stable matching!