Lecture 5 – Stable matching problem
Reading: KT Chapter 1

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Stable Matching Problem

• Q. Is assignment X-A, Y-B, Z-C stable?
• A. Yes.

<table>
<thead>
<tr>
<th>Residents’ Preference Profile</th>
<th>Hospitals’ Preference Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>favorite</strong></td>
<td><strong>least favorite</strong></td>
</tr>
<tr>
<td>1st</td>
<td>2nd</td>
</tr>
<tr>
<td>Xavier</td>
<td>Amy</td>
</tr>
<tr>
<td>Yancey</td>
<td>Bertha</td>
</tr>
<tr>
<td>Zeus</td>
<td>Amy</td>
</tr>
</tbody>
</table>
Propose-And-Reject Algorithm

- The [Gale-Shapley 1962] deferred acceptance algorithm is an intuitive method that guarantees to find a stable matching.

Initially all $m \in M$ and $w \in W$ are free
While there is a man $m$ who is free and hasn't proposed to every woman
    Choose such a man $m$
    Let $w$ be the highest-ranked woman in $m$’s preference list to whom $m$ has not yet proposed
    If $w$ is free then
      $(m,w)$ become engaged
    Else $w$ is currently engaged to $m'$
      If $w$ prefers $m'$ to $m$ then
        $m$ remains free
      Else $w$ prefers $m$ to $m'$
        $(m,w)$ become engaged
        $m'$ becomes free
    Endif
Endif
Endwhile
Return the set $S$ of engaged pairs

So far...

- Stable matching problem.
  - Given $n$ men and $n$ women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm.

- Q. Does it correctly find a stable matching for any problem instance?

- Q. How to implement GS algorithm efficiently?

- Q. If there are multiple stable matchings, which one does GS find?
Proof of Correctness: Termination

• Observation 1. Men propose to women in decreasing order of preference.

• Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

• Claim. Algorithm terminates after at most \( n^2 \) iterations of while loop.
• Pf. Each time through the while loop a man proposes to a new woman. There are only \( n^2 \) possible proposals.

\[ n(n-1) + 1 \text{ proposals required} \]

Proof of Correctness: Perfection

• Claim. All men and women get matched.

• Pf. (by contradiction)
  • Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
  • Then some woman, say Amy, is not matched upon termination.
  • By Observation 2, Amy was never proposed to.
  • But, Zeus proposes to everyone, since he ends up unmatched.
Proof of Correctness: Stability

- Claim. No unstable pairs in a matching $S^*$.  
- Pf. (by contradiction)
  - Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching $S^*$.  
  - Case 1: Z never proposed to A.  
    $\Rightarrow$ Z prefers his GS partner to A.  
    $\Rightarrow$ A-Z is stable.  
  - Case 2: Z proposed to A.  
    $\Rightarrow$ A rejected Z (right away or later)  
    $\Rightarrow$ A prefers her GS partner to Z.  
    $\Rightarrow$ A-Z is stable.  
  - In either case A-Z is stable, a contradiction.  

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?  
  - An instance with two stable matchings.  
    - A-X, B-Y, C-Z.  
    - A-Y, B-X, C-Z.  

\[
\begin{array}{|c|c|c|}
\hline
\text{1st} & \text{2nd} & \text{3rd} \\
\hline
\text{Xavier} & A & B & C \\
\hline
\text{Yancey} & B & A & C \\
\hline
\text{Zeus} & A & B & C \\
\hline
\end{array}
\]
\[
\begin{array}{|c|c|c|}
\hline
\text{1st} & \text{2nd} & \text{3rd} \\
\hline
\text{Amy} & Y & X & Z \\
\hline
\text{Bertha} & X & Y & Z \\
\hline
\text{Clare} & X & Y & Z \\
\hline
\end{array}
\]
Understanding the Solution

• Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

• Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

• Man-optimal assignment. Each man receives best valid partner.

• Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

• Claim. All executions of GS yield woman-pessimal assignment, which is a stable matching!