Stable Matching Problem

- Q. Is assignment X-A, Y-B, Z-C stable?
- A. Yes.

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/
Propose-And-Reject Algorithm

- The [Gale-Shapley 1962] deferred acceptance algorithm is an intuitive method that guarantees to find a stable matching.

<table>
<thead>
<tr>
<th>Initially all $m \in M$ and $w \in W$ are free</th>
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<tbody>
<tr>
<td>While there is a man $m$ who is free and hasn't proposed to every woman</td>
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<tr>
<td>Choose such a man $m$</td>
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<tr>
<td>Let $w$ be the highest-ranked woman in $m$‘s preference list to whom $m$ has not yet proposed</td>
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<tr>
<td>If $w$ is free then</td>
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<td>$(m, w)$ become engaged</td>
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<tr>
<td>Else $w$ is currently engaged to $m$</td>
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<tr>
<td>If $w$ prefers $m'$ to $m$ then</td>
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<tr>
<td>$m$ remains free</td>
</tr>
<tr>
<td>Else $w$ prefers $m$ to $m'$</td>
</tr>
<tr>
<td>$(m, w)$ become engaged</td>
</tr>
<tr>
<td>$m'$ becomes free</td>
</tr>
<tr>
<td>Endif</td>
</tr>
<tr>
<td>Endif</td>
</tr>
<tr>
<td>Endwhile</td>
</tr>
<tr>
<td>Return the set $S$ of engaged pairs</td>
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</tbody>
</table>

So far...

- **Stable matching problem.**
  - Given $n$ men and $n$ women, and their preferences, find a stable matching if one exists.

- **Gale-Shapley algorithm.**

- **Q.** Does it correctly find a stable matching for any problem instance?

- **Q.** How to implement GS algorithm efficiently?

- **Q.** If there are multiple stable matchings, which one does GS find?
Proof of Correctness: Termination

- **Observation 1.** Men propose to women in decreasing order of preference.

- **Observation 2.** Once a woman is matched, she never becomes unmatched; she only "trades up."

- **Claim.** Algorithm terminates after at most \( n^2 \) iterations of while loop.
- **Pf.** Each time through the while loop a man proposes to a new woman. There are only \( n^2 \) possible proposals.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
1^\text{st} & 2^\text{nd} & 3^\text{rd} & 4^\text{th} & 5^\text{th} \\
\hline
\text{Wyatt} & B & C & A & E & D \\
\text{Kaiser} & C & D & A & B & E \\
\text{Vanary} & D & A & B & C & E \\
\text{Zeus} & A & B & C & D & E \\
\hline
\text{Amy} & W & X & E & Z & V \\
\text{Barth} & X & Y & Z & V & W \\
\text{Clare} & Y & Z & W & X & Y \\
\text{Diane} & Z & V & W & X & Y \\
\text{Erika} & V & W & Z & Y & X \\
\end{array}
\]

\( n(n - 1) + 1 \) proposals required

Proof of Correctness: Perfection

- **Claim.** All men and women get matched.

- **Pf.** (by contradiction)
  - Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2, Amy was never proposed to.
  - But, Zeus proposes to everyone, since he ends up unmatched.
Proof of Correctness: Stability

• Claim. No unstable pairs in a matching $S^\ast$.
• Pf. (by contradiction)
  • Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching $S^\ast$.
  
  • Case 1: Z never proposed to A.
    $\Rightarrow$ Z prefers his GS partner to A.
    $\Rightarrow$ A-Z is stable.

  • Case 2: Z proposed to A.
    $\Rightarrow$ A rejected Z (right away or later)
    $\Rightarrow$ A prefers her GS partner to Z.
    $\Rightarrow$ A-Z is stable.

• In either case A-Z is stable, a contradiction. $\blacksquare$

Let’s discuss implementation
Understanding the Solution

• Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

• An instance with two stable matchings.
  • A-X, B-Y, C-Z.
  • A-Y, B-X, C-Z.

Understanding the Solution

• Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

• Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

• Man-optimal assignment. Each man receives best valid partner.

• Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

• Claim. All executions of GS yield woman-pessimal assignment, which is a stable matching!