CS231 Algorithms
Prof. Lyn Turbak
Wellesley College

Handout #5
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ASYMPTOTICS AND FUNCTIONS

Note: This handout summarizes highlights of CLR Chapter 2. See the book for more details.

Motivation

Fine-grained bean counting exposes too much detail for comparing functions.

Want a course-grained way to compare functions that ignores constant factors and focuses on their relative growth in the limit as input sizes get large.

For example, consider:

	n = 1	n = 1,000	n = 1,000,000
$p(n) = 100n + 1000$			
$q(n) = 3n^2 + 2n + 1$			
$r(n) = 0.1n^2$			

Sketch the above functions on the same set of axes:

How Do Your Functions Grow?

Asymptotic notation is a way of characterizing functions that facilitates comparing their growth in the limit of large inputs. Here is an informal summary of the notation:

Notation	Pronunciation	Loosely
$f \quad \omega(g)$	f is way bigger than g	$f > g$
$f \quad \Omega(g)$	f is at least as big as g	$f \geq g$
$f \quad \Theta(g)$	f is about the same as g	$f = g$
$f \quad O(g)$	f is at most as big as g	$f \leq g$
$f \quad o(g)$	f is way smaller than g	$f < g$

Notes:

- Each of $\omega(g)$, $\Omega(g)$, $\Theta(g)$, $O(g)$, $o(g)$ denotes a *set* of functions. Thus, $\omega(g)$ is the set of all functions way bigger than g, $\Omega(g)$ is the set of all functions at least as big as g, etc.

- The notation $f = \omega(g)$ is really shorthand for $f \quad \omega(g)$.

- The phrases “is at least O(...)” and “is at most Ω(...)” are non-sensical. “Is at least” should be written Ω, and “is at most” should be written O.

Intuitively, what are the relationships between p, q, and r?

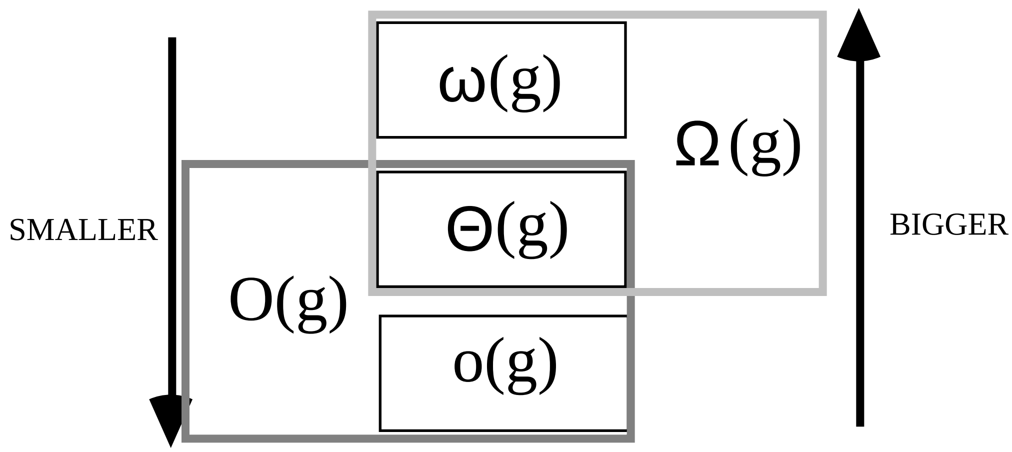
Relating the Notations

Here are some of the relationships between the notations:

If f ω(g), then f Ω(g).
If f o(g), then f O(g).
Ω(g) ω(g) Θ(g)
O(g) o(g) Θ(g)
Θ(g) = Ω(g) ∩ O(g)
f ω(g) if and only if g o(f)
f Ω(g) if and only if g O(f)
f Θ(g) if and only if g Θ(f)

Warning: unlike numbers, not every pair of functions is comparable!

The following diagram depicts some of these relationships:



Formalizing o and ω

$f \quad o(g)$ if $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$

$f \quad \omega(g)$ if $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$

Examples:

Show p   o(r)

Show r   ω(r)

Formalizing O, Ω, and Θ

$O(g) = \{f \mid$ there exist positive constants c, n_0 such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0$.$\}$

Think of this as a game. Suppose you claim that f   O(g). Then you select c and n_0 , but I try to find a particular n that defeats your claim.

$\Omega(g) = \{f \mid$ there exist positive constants c, n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0$.$\}$

$\Theta(g) = \{f \mid$ there exist positive constants c_1 , c_2 , n_0 such that
$0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0$.$\}$

Examples

Show p   O(q)

(1) use c = 1, n_0= 1000

(2) use c = 1000, n_0= 1

Can we show q_ O(p) ?

Show r_ O(q) (use c = 1 n_0= 1)

Show q   O(r) (use c = 40; what must n_0 be?)

If $\lim_{n \to \infty}$ (f(n)/g(n)) = k > 0, then f   Θ (g). This is an easy way to show that two functions are related by Θ. E.g., use it to show that q   Θ (r) and r   Θ (q).

The converse of the above limit trick is not true. That is, although the limit trick works most of the time to show that two functions are related by Θ, there are some cases where it doesn't work. E.g., f(n) = 2 + sin(n) and g(n) = 2.

Does anything Fall Between the Cracks?

The diagram on p. 3 implies that there are functions that are O(g) that are neither o(g) nor Θ (g), and there are functions that are Ω(g) that are neither ω(g) nor Θ (g). Here's a concrete example:

$f(n) = 1/n$
$g(n) = n$
$h(n) = n^{\sin(n)}$

Show that h O (g) , but h o (g) and h Θ (g). (Similarly, h Ω (f), but h ω (g) and h Θ (g)).

Incomparable Functions

Not every two functions are comparable. Is k(n) = sqrt(n) related to h(n) above?

Exponentials

Notation:

- a^n = the product of *n* copies of *a*.
- $a^{-n} = \frac{1}{a^n}$

Key Identities:

- $a^m a^n = a^{m+n}$ {Special case: $a^0 = 1$.}
- $(a^m)^n = a^{mn} = (a^n)^m$

Examples:

$(5^2)^3 =$

$5^2 \ 5^3 =$

$5^2 + 5^3 =$

$25^{(\frac{3}{2})} =$

Let:

$f(n) = 2^n$

$g(n) = 3^n$

$h(n) = 2^{cn}$

$k(n) = 2^{c+n}$

What symbols can fill the following blanks?

g _____(f)

h _____(f) (c < 1)

h _____(f) (c = 1)

h _____(f) (c > 1)

k _____(f) (any c)

Logarithms

Notation:

- $\log_b a$ = the power to which b must be raised to equal a. (More loosely, it is the number of times that a can be divided by b to reach 1.)
- $\log_b (1/a) = -\log_b a$
- $\lg n = \log_2 n$
- $\ln n = \log_e n$
- $\lg^k(n) = (\lg n)^k$

Key Identities (duals of exponential identities):

- $\log_c(ab) = \log_c(a) + \log_c(b)$ {Special case: $\log_c 1 = 0$}

  {Special case: $\log_c a^n = n \log_c a$}
- $\log_c(a) = \log_c(b) \quad \log_b(a)$

Examples:

$\lg(2n^3) =$

$\ln(32) =$

Relating Exponentials and Logarithms

Key Identities:

- $b^{(\log_b a)} = a = \log_b(b^a)$

Examples:

$\lg \sqrt[3]{4} =$

$32^{(\lg n)} =$

Asymptotics Involving Exponentials and Logarithms

How do $\log_2 n$ and $\log_3 n$ compare?

How do 2^n and 3^n compare?

Fact 1: if a > 0, $\lim_{n \to \infty} \left(\frac{a^n}{n^b} \right) = \infty$

Fact 1 implies $a^n \quad \omega(n^b)$.
In other words: *Any positive exponential grows faster than any polynomial.*

Substituting lg n for n and 2^a for a in Fact 1 yields:

Fact 2: if a > 0, $\lim_{n \to \infty} \left(\frac{n^a}{\lg^b n} \right) = \infty$

Fact 2 implies $n^a \quad \omega(\lg^b n)$.
In other words: *Any positive polynomial grows faster than any polylogarithmic function.*

Factorials

Definition: $n! = 1 \; 2 \; 3 \ldots \; n$

Stirling's approximation: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$

Asymptotics derivable from Stirling's approximation:

- $n! = o(n^n)$
- $n! = \omega(2^n)$
- $\lg(n!) = \Theta(n \lg n)$
