

**COUNTING AND PROBABILITY**

**Reading:** CLR Sections 6.1 -- 6.3; 6.6.

**Counting**

Given a set  $S$  with  $n$  elements, how many ways are there to choose  $k$  elements from  $S$ ? The answer depends on whether (1) order matters and (2) duplicates are allowed.

	<b>Duplicates not allowed (<math>k \leq n</math>)</b>	<b>Duplicates allowed (<math>k</math> may be <math>&gt; n</math>)</b>
<b>Order matters (sequences)</b>	<i>k-permutation</i> $\frac{n!}{(n - k)!}$	<i>k-tuple (k-string)</i> $n^k$
<b>Order doesn't matter (sets)</b>	<i>k-combination</i> $\frac{n!}{k! (n - k)!}$	<i>k-selection</i> $\frac{(n - 1 + k)!}{k! (n - 1)!}$

Example:  $S = \{a, b, c, d\}$ ,  $k = 2$

Type	Elements	Number
2-permutations of $S$		
2-combination of $S$		
2-strings of $S$		
2-selections of $S$		

Exercise: Try  $k = 3$



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## Events

An **event** is a subset of a sample space.

Experiment 1:

Event 1a: First flip is a head =

Event 1b: Second flip is a tail =

Event 1c: Exactly two tails =

Event 1d: Two consecutive flips the same =

Experiment 2:

Event 2a: First flip is a head =

Event 2b: First flip is a tail =

Event 2c: Even number of flips =

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## Event Combinators

If A and B are events in sample space S,

"A and B" is translated " $A \cap B$ "

"A or B" is translated " $A \cup B$ "

"not A" is translated " $S - A$ "

Two events A and B are **mutually exclusive** if  $A \cap B = \emptyset$ .

Of the four events in Experiment 1, which pairs are mutually exclusive?

Of the three events in Experiment 2, which pairs are mutually exclusive?

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## Probability Distributions

A **probability distribution**  $\Pr\{\cdot\}$  on a sample space  $S$  is any mapping from events to  $[0..1]$  such that the following axioms hold:

1.  $\Pr\{A\} \geq 0$  for any event  $A$ .
2.  $\Pr\{S\} = 1$
3. If  $A \cap B = \emptyset$  then  $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$

Some useful theorems:

$$\Pr\{\emptyset\} = 0.$$

$$\Pr\{S - A\} = 1 - \Pr\{A\}.$$

$$\text{If } A \subset B \text{ then } \Pr\{A\} \leq \Pr\{B\}$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

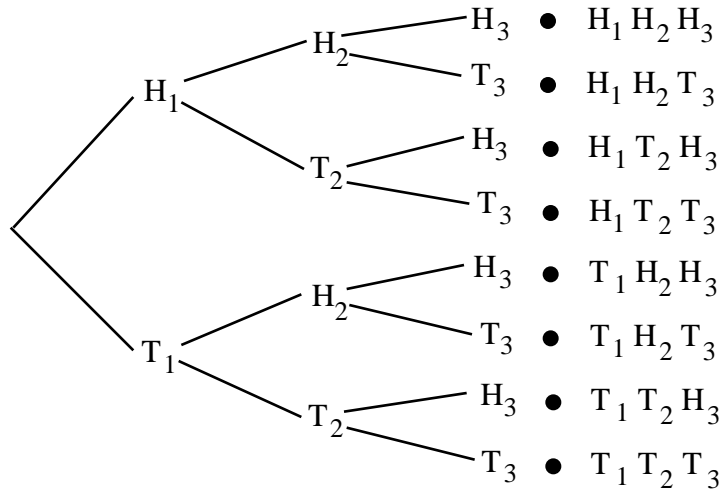
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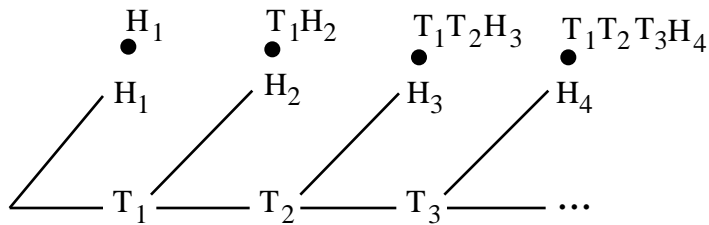
## Probability Trees

A sequential sample space gives rise to a **probability tree**. If events at distinct stages of tree are independent (defined below), probability of leaf is product of probabilities on path to leaf.

Experiment 1 with  $H_1 = (1/3)$ ,  $H_2 = (1/4)$ ,  $H_3 = (1/5)$



Experiment 2 with  $H_i = (1/3)$



The probability of an event in a sequential sample space is the sum of the probability of its member leaves in the probability tree. What are the probabilities of the following events?

1a:

1b:

1c:

1d:

2a:

2b:

2c:

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### **Birthday Paradox**

How many people must be in a room before there is a 50% chance that two of them have the same birthday? A 90% chance? A 99% chance? (Assume that for any given person, each day of the year is equally likely as a birthday. Ignore leap years.)



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## Conditional Probability

$$\text{Probability of A given B} = \Pr\{A | B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

*Example 1:* In Experiment 1 with  $H_1 = (1/3)$ ,  $H_2 = (1/4)$ ,  $H_3 = (1/5)$ , what is probability of 1c (exactly two tails) given 1d (two consecutive flips the same)?

*Example 2:* Ann has two children, one of which is a girl. What is the probability that the other child is a boy? (Assume a child is girl or boy with equal probability.)

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## Independence

Events A and B are **independent** if  $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$ .  
(Equivalently,  $\Pr\{A | B\} = \Pr\{A\}$ )

Of the four events in Experiment 1, which pairs are independent?

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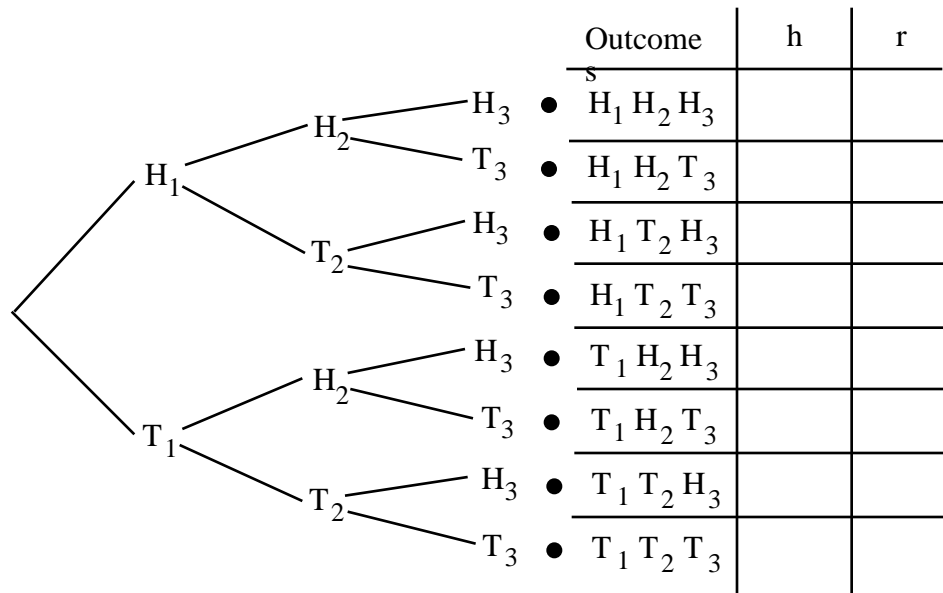
## Discrete Random Variables

A discrete random variable on a sample space  $S$  is a function  $f: S \rightarrow \text{Real}$ .

E.g., in Experiment 1 let

$h(x)$  = number of heads in outcome  $x$ .

$r(x)$  = length of longest run in outcome  $x$ .



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## Probability Density Function

The **pre-image** of a value  $v$  under  $f = f^{-1}(v) = \{x \in S \mid f(x) = v\}$   
(This differs from CLR's notation " $f = v$ ", which I find confusing.)

Each pre-image is an event:

$$h^{-1}(1) =$$

$$r^{-1}(3) =$$

The **probability density function (PDF)** of discrete random variable  $f$  is another function that maps each target value  $v$  of  $f$  to its probability. This is determined by summing the probabilities of all sources  $x$  that  $f$  maps to  $v$ :

$$[\text{PDF}(f)](v) = \Pr\{f^{-1}(v)\} = \sum_{\{x \in S \mid f(x) = v\}} \Pr\{x\}$$

What is the function  $\text{PDF}(h)$ ?

What is the function  $\text{PDF}(r)$ ?

Discrete random variables  $f$  and  $g$  are **independent** if for all  $w$  and  $v$ , the events  $f^{-1}(w)$  and  $g^{-1}(v)$  are independent.

Are  $h$  and  $r$  independent?

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## Expected Value

The **expected value**  $E[f]$  of a discrete random variable  $f$  is a weighted average in which target values  $v$  of  $f$  are weighted by their probability:

$$E[f] = \sum_v v \Pr\{f^{-1}(v)\}$$

Examples:

Expected number of heads in Experiment 1 =  $E[h] =$

Expected length of runs in Experiment 2 =  $E[r] =$

Expected number of flips in Experiment 2 =

Expectations are **linear**. That is, if  $f$  &  $g$  are discrete random variables and  $c$  is a constant:

$$E[f + g] = E[f] + E[g] \quad \{\text{The sum of functions } f \text{ and } g \text{ is a function } (f + g)(x) = f(x) + g(x).\}$$

$$E[cf] = c[E[f]] \quad \{\text{The scaling by } c \text{ of a function } f \text{ is a function } (cf)(x) = cf(x).\}$$

If  $f$  and  $g$  are independent, then

$$E[fg] = E[f]E[g] \quad \{\text{The product of functions } f \text{ and } g \text{ is a function } (fg)(x) = f(x)g(x).\}$$

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