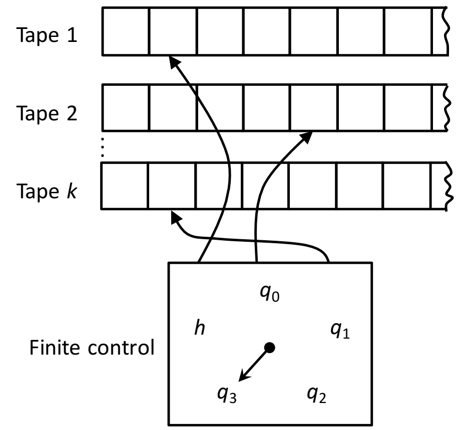
Building a Better Mousetrap

Sipser: Section 3.2 pages 176 - 182

Multitape Turing Machines

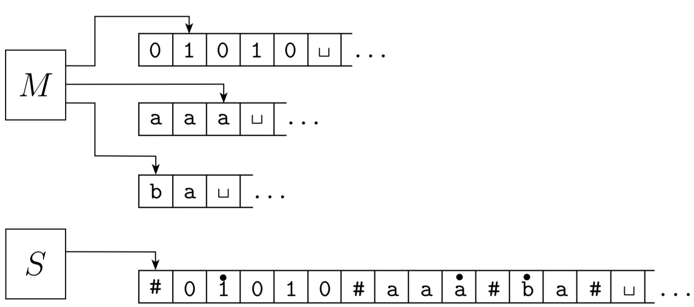


Formally, we need only change the transition function to

$$\delta\colon Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{L, R\}^k$$

Evidence of Turing Robustness

Theorem. Every multitape Turing machine has an equivalent single tape Turing machine.



Corollary. A language is Turing-recognizable if and only if some multitape Turing machine recognizes it.

Recognizing *Composite* Numbers

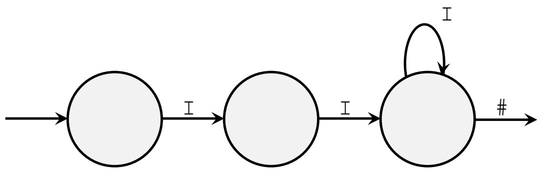
- Let $L = \{ 1^n : n \text{ is a composite number} \}$.
- Designing a Turing machine to accept L would seem to involve factoring n.
- However, if we could guess …

Guessing Games

Design a machine M that on input I^n performs the following steps:

1. Nondeterministically choose two numbers p, $q > 1$ and transform the input into $\#I^n\#I^p\#I^q\#$.
2. Multiplies p by q to obtain $\#I^n\#I^{pq}\#$.
3. Checks the number of I's before and after the middle $\#$ for equality. Accepts if equal, and rejects otherwise.

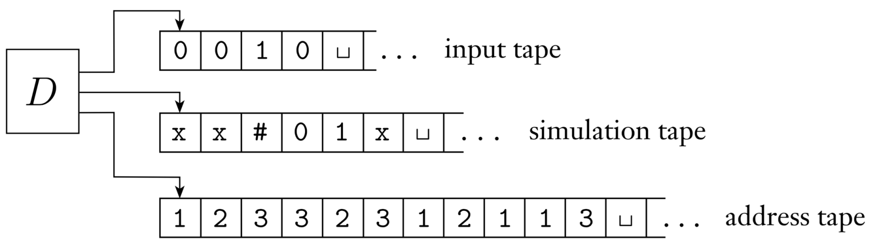
The Guessing Machine



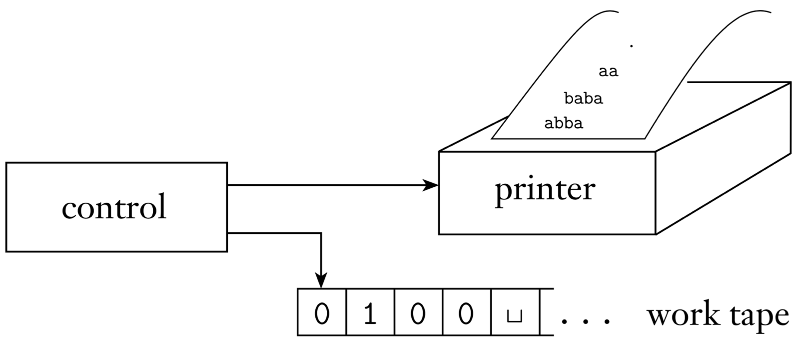
Again, the only difference between this variant and the standard TM is the transition function: $\delta: Q \times \Gamma \longrightarrow P(Q \times \Gamma \times \{L, R\})$

Guessing Doesn't Help

Theorem. Every nondeterministic Turing machine has an equivalent deterministic Turing machine.



Recursively Enumerable



Enumerators

Theorem. A language is Turing-recognizable if and only if some enumerator enumerates it.

Proof. ($\Leftarrow$) Suppose enumerator E enumerates L.

Define M = "On input w:

Run E. Every time E outputs a string, compare it with w.

If w ever appears in the output of E, *accept*."

Recursively Enumerable

Theorem. A language is Turing-recognizable if and only if some enumerator enumerates it.

Proof. ($\Rightarrow$) Suppose TM M recognizes L. Build a lexicographic enumerator to generate the list of all possible strings s_1, s_2, … over Σ.

Define E = "Ignore input.

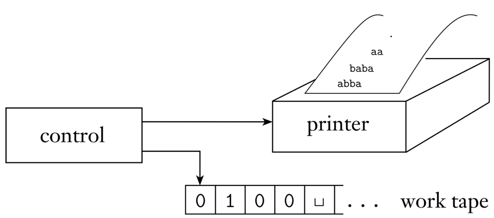
Repeat the following for i = 1, 2, 3, …

Run M for i steps on each of $s_1, s_2, \ldots, s_i$.

If any computation accepts, print corresponding s_j."

TM's Take Their Own Sweet Time

- Recognizers, like enumerators, may take a while to answer *yes*, … and even longer to answer *no*.

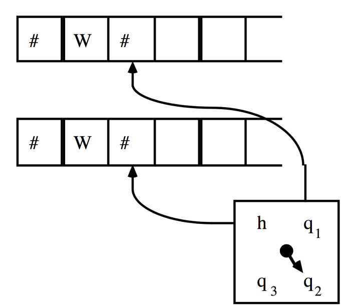


- A TM that halts on all inputs is called a decider. A ***decider*** that recognizes a language is said to ***decide*** that language.
- Call a language ***Turing-decidable*** if some Turing machine decides it.

Recognizable versus Decidable

Theorem. A language is *Turing-decidable* if and only if both it and its complement are *Turing-recognizable*.

Proof. ($\Rightarrow$) By definition.



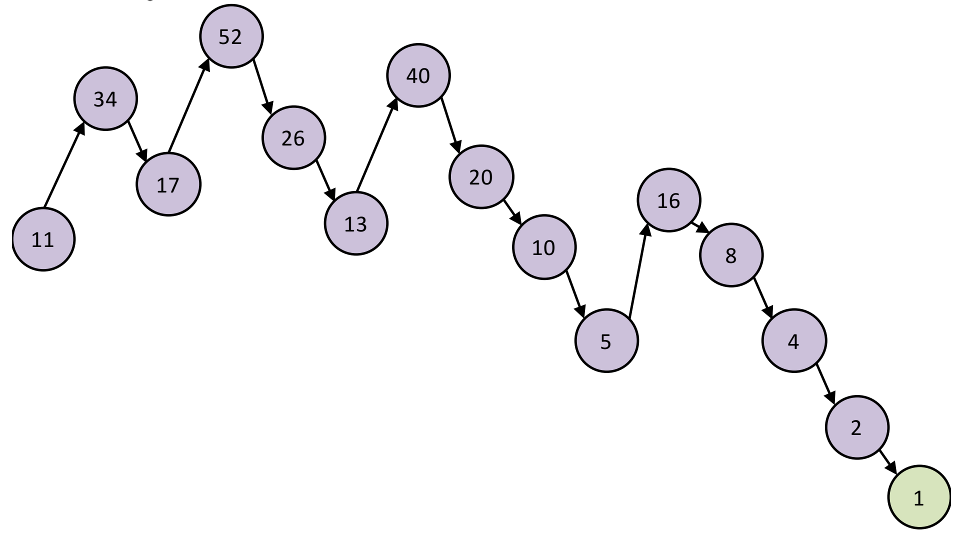
($\Leftarrow$) Simulate, in parallel, M_L on tape 1 and $M_{L'}$ on tape 2.

The Hailstone Sequence

```
HailstoneSequence(n)
      if (n ≠ 1)
            if (n is even)
                  HailstoneSequence(n/2)
            else
                  HailstoneSequence(3*n+1)
```

Given an integer $n > 0$, does this process terminate?

Example Sequence, $n = 11$



Hailstone Turing Machine

Let $H = \{ 1^n \mid n > 0$ and the hailstone sequence terminates for $n \}$.

We construct TM M to recognize language H.

M = "On input w:
1. If the input is ε, *reject*.
2. If the input has length 1, *accept*.
3. If the input has even length, halve its length.
4. If the input has odd length, triple its length and append 1.
5. Go to stage 2."

The Simplest Impossible Problem

- Is is unknown whether this process will terminate for all natural numbers.
- It is unknown whether TM M might loop forever.