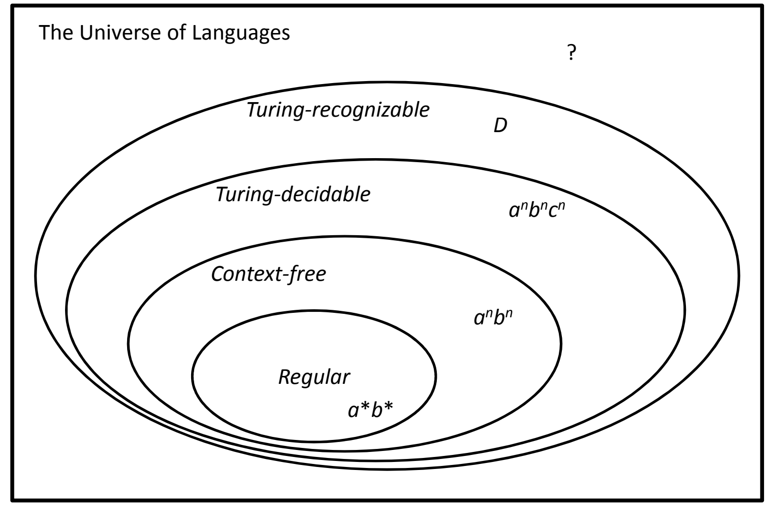
What Machines Cannot Do

Sipser: Section 4.2 pages 201 - 210

Will This Ever End?



The Sizes of Sets

- Comparing the sizes of two finite sets is easy
- Do all infinite sets have the same size? How can we compare the relative sizes of two infinite sets?

$\mathbb{W}$	$\mathbb{N}$	$\mathbb{E}$
0	1	2
1	2	4
2	3	6
3	4	8
4	5	10
5	6	12
…	…	…

The Sizes of Sets

- Two sets have the same size if the elements of one set can be paired with the elements of the other set.
- A function that is both one-to-one and onto is called a ***correspondence*** (bijection). Two sets have the same size if there is a correspondence between them.

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*A set is **countable** if either it is finite or it has the same size as $\mathbb{N}$.*

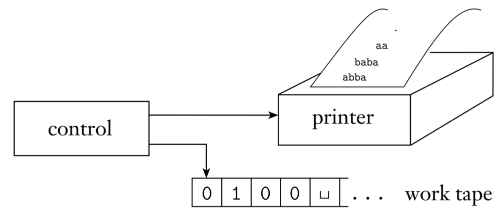
ℝ is uncountable (proof by diagonalization)

- We show that no correspondence exists between ℕ and ℝ.
- To reach a contradiction, suppose that a correspondence *f* does exist between ℕ and ℝ.
- We will find *x* in ℝ that is not paired with anything in ℕ, which will be our contradiction.

n	*f*(*n*)
1	3.14159…
2	55.55555…
3	0.12345…
4	0.50000…
5	1.414213…
…	…

Finite Representation of Languages

- A finite representation of a language must itself be a string over some alphabet Σ. Furthermore, different languages must have distinct representations.



- How many strings can we represent over any given alphabet?

How Many is Many?

Theorem. Let Σ be any finite alphabet containing at least one element. The set of all strings Σ* over Σ is countably infinite.

How Many Languages?

Definition. Let 2^{Σ^*}, known as the power set of Σ*, be the set of all subsets of Σ*, i.e., the set of all languages over Σ.

Theorem. The set 2^{Σ^*} is uncountable.

Proof. For each language $A \in 2^{\Sigma^*}$, create a unique infinite binary sequence.

Σ* = {	ε,	0,	1,	00,	01,	10,	11,	000,	001,	…	}
A = {		0,		00,	01,			000,	001,	…	}
f(*A*) =	0	1	0	1	1	0	0	1	1	…	

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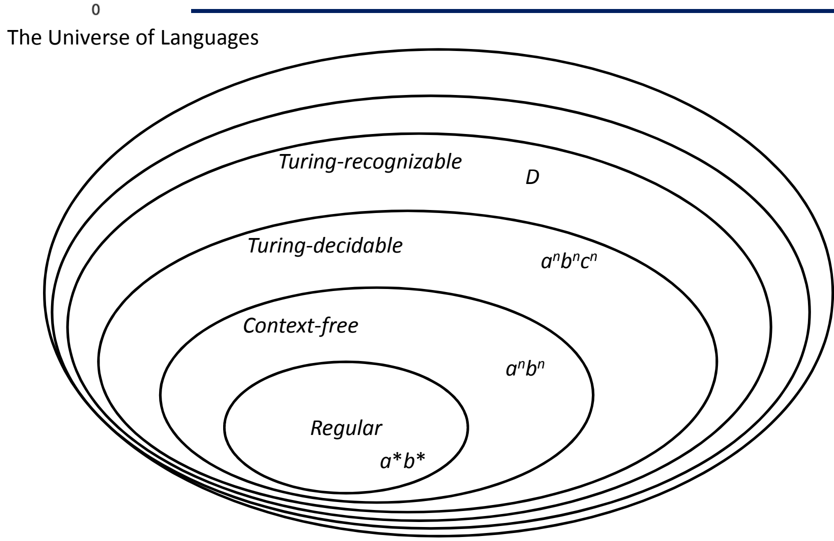
$f(A) =$	1	0	0	0	1	0	1	1	1	...

Thus, we have a correspondence f between 2^{Σ^*} and infinite binary sequences. Since the set of infinite binary sequences is uncountable (see homework), so is 2^{Σ^*}.

The Sad Conclusion…



The Trick is to Get all the Good Ones

Algorithm	=	*Turing Machine*

Let's Try This One*

Definition. $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

* By analogy with our old friends A_{DFA} and A_{CFG}.

A_{TM} is Turing-Recognizable

U = "On input $\langle M, w \rangle$, where M is a TM and w a string:

1. Simulate M on input w.

2. If M ever enters its accept state, *accept*. If M ever enters its reject state, *reject*."

The *universal Turing machine.*

The Halting Problem

We could use U to decide A_{TM} if we had some way to determine whether M would halt on input w.

"On input $\langle M, w \rangle$, where M is a TM and w a string:

1. Determine whether M on input w will ever halt. If not, then *reject*.

2. Otherwise, simulate M on input w.

3. If M enters its accept state, *accept*. If M enters its reject state, *reject*."

Some People Don't Know When to Stop

Theorem. $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$ is undecidable.

Proof. Suppose TM H decides A_{TM}. That is,

$$H(\langle M, w \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Calling H as a Subroutine

Define the contrary TM D:

D = "On input $\langle M \rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M \rangle \rangle$.*
2. Output the opposite of what H outputs.

That is,

$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

* Think of a Python compiler written in Python.

Calling D on Itself

$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

$\bar{A}_{TM}$ is not even Turing-recognizable

Corollary. $\bar{A}_{TM}$ is not Turing-recognizable.

Proof. If so, then both A_{TM} and $\bar{A}_{TM}$ would be Turing-recognizable. But, then …

Out of Bounds