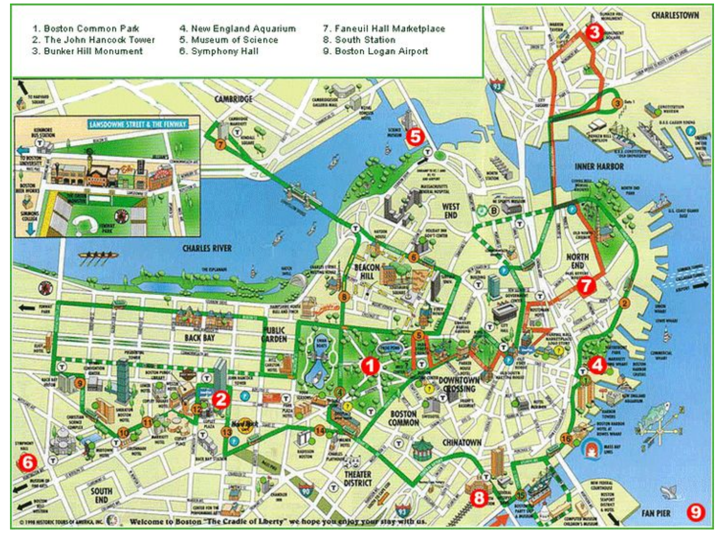
Undecidable Problems About Languages

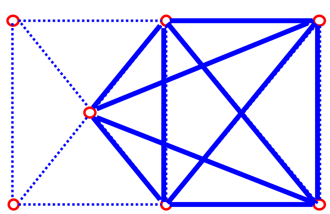
Sipser: Section 5.1 pages 215 - 226

Reducibility



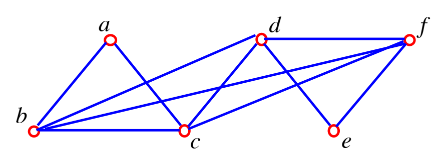
Clique and Independent Set

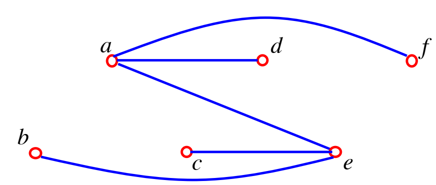
CLIQUE = {<*G*,*k*> | *G* is a graph with a *k*-clique}



INDEPENDENT = {<*G*,*k*> | *G* is a graph containing an independent set of size *k*}

CLIQUE reduces to INDEPENDENT





Certified Impossible

Theorem. A_{TM} = { $\langle M, w \rangle$ | *M* is a TM and *M* accepts *w* } is undecidable.

Definition. $HALT_{TM}$ = { $\langle M, w \rangle$ | *M* is a TM and *M* halts on input *w* }

The Halting Problem (Again!)

Theorem. $HALT_{TM}$ is undecidable.

Proof Idea. We know A_{TM} is undecidable. We need to reduce one of $HALT_{TM}$ or A_{TM} to the other.

Which way to go?

$HALT_{TM}$ is undecidable

Proof. Suppose *R* decides $HALT_{TM}$. Define

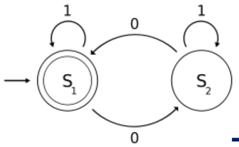
S = "On input $\langle M, w \rangle$, where *M* is a TM and *w* a string:

1. Run TM *R* on input $\langle M, w \rangle$.
2. If *R* rejects, then *reject*.
3. If *R* accepts, simulate *M* on input *w* until it halts.
4. If *M* enters its accept state, *accept*. If *M* enters its reject state, *reject*."

Does *M* Accept Anything at All?

Definition. E_{TM} = { $\langle M \rangle$ | *M* is a TM and $L(M) = \varnothing$ }

Theorem. E_{TM} is undecidable.



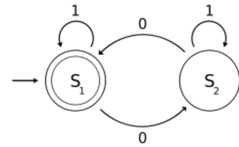
E_{TM} = {<M> | M is a TM & $L(M) = \varnothing$}

Proof. Given an input <M, w> we construct a machine M_w as follows:

M_w = "On input x:

1. If $x \neq w$, *reject*.
2. If $x = w$, run M on input w and *accept* if M does."

to be continued …



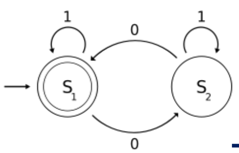
The Proof Continues

Proof continued.

Suppose TM R decides E_{TM}. Define

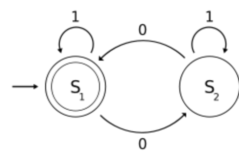
S = "On input <M, w>:

1. Use the description of M and w to construct M_w.
2. Run R on input <M_w>.
3. If R accepts, *reject*. If R rejects, *accept*."



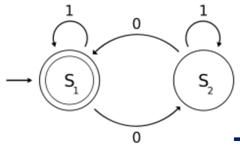
With Power Comes Uncertainty

	M accepts w	$L(M) = \varnothing$	$L(M_1) = L(M_2)$
Turing machines			
Pushdown machines			
Finite machines			



It's Even Worse Than You Thought

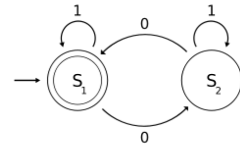
Rice's Theorem. Any *nontrivial property* of the languages recognized by Turing machines is undecidable.



For Example

Definition. $REGULAR_{TM}$ = { <*M*> | *M* is a TM and $L(M)$ is regular}.

Theorem. $REGULAR_{TM}$ is undecidable.



$REGULAR_{TM}$ is undecidable

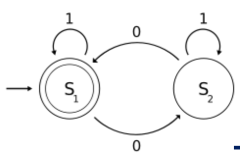
Proof. Let *R* be a TM that decides $REGULAR_{TM}$. Define

S = "On input <*M*, *w*>:

1. Construct TM

   M_2 = "On input *x*:

   1. If *x* has the form 0^n1^n, *accept*.
   2. Otherwise, run *M* on input *w* and *accept* if *M* accepts *w*.
2. Run *R* on input <M_2>.
3. If *R* accepts, *accept*. Otherwise, if *R* rejects, *reject*."



Problems

- Let EQ_{TM} = { <M_1, M_2> | M_1 and M_2 are TMs and $L(M_1) = L(M_2)$ }. Show that EQ_{TM} is undecidable by reducing E_{TM} to EQ_{TM}.

- Consider the problem of determining whether a two-tape TM ever writes a nonblank symbol on its second tape when run on input *w*. Formulate this problem as a language and show that it is undecidable. (Hint: create an intermediary TM *T* that writes a nonblank symbol on its second tape iff *M* accepts *w*.)