**Mapping Reducibility**

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**Theorem.** $E_{TM}$ is undecidable.

**Proof.** Given an input $<M, w>$ we construct a machine $M_w$ that accepts a nonempty language iff $M$ accepts $w$:

$M_w = \text{"On input } x:\n 1. \text{ If } x \neq w, \text{ reject.}\n 2. \text{ If } x = w, \text{ run } M \text{ on input } w \text{ and accept if } M \text{ does."}$

Suppose TM $R$ decides $E_{TM}$ and establish a contradiction by creating a decider $S$ of $A_{TM}$:

$S = \text{"On input } <M, w>:\n 1. \text{ Use the description of } M \text{ and } w \text{ to construct } M_w.\n 2. \text{ Run } R \text{ on input } <M_w>.\n 3. \text{ If } R \text{ accepts, reject. If } R \text{ rejects, accept."}$

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**Computable Functions**

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**Definition.** A function $f : \Sigma^* \rightarrow \Sigma^*$ is a computable function if some Turing machine $M$, on every input $w$, halts with just $f(w)$ on its tape.

**Example.** The increment function

$\text{inc}^+: \{1\}^* \rightarrow \{1\}^*$

is Turing computable.
Machine Transformers

\[ F = \text{"On input } \langle M \rangle:\]
1. Construct the machine
   \[ M_\infty = \text{"On input } x:\]
   1. Run \( M \) on \( x \).
   2. If \( M \) accepts, accept.
   3. If \( M \) rejects, loop.
2. Output \( \langle M_\infty \rangle \)."

Mapping Reducibility

**Definition.** Language \( A \) is mapping reducible to language \( B \), written \( A \leq_m B \), if there is a computable function \( f: \Sigma^* \rightarrow \Sigma^* \), where for every \( w \),
\[ w \in A \iff f(w) \in B. \]

Problem Reduction

**Theorem.** If \( A \leq_m B \) and \( B \) is decidable, then \( A \) is decidable.

The Contapositive is Also Useful

**Theorem.** If \( A \leq_m B \) and \( B \) is decidable, then \( A \) is decidable.

**Corollary.** If \( A \leq_m B \) and \( A \) is undecidable, then \( B \) is undecidable.
Similarly ...

**Theorem.** If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

**Corollary.** If $A \leq_m B$ and $A$ is not Turing-recognizable, then $B$ is not Turing-recognizable.

A Familiar Mapping Reduction

$A_{TM} = \{ <M, w> | M \text{ is a TM and } M \text{ accepts } w \}$

$\leq_m$

$HALT_{TM} = \{ <M, w> | M \text{ is a TM and } M \text{ halts on input } w \}$

Solvable, Half-Solvable, Out-to-Lunch

$F = \text{"On input } <M, w>:\$

1. Construct the machine $M_{\infty} = \text{"On input } x:"
   1. Run $M$ on $x$.
   2. If $M$ accepts, accept.
   3. If $M$ rejects, loop.
2. Output $<M_{\infty}, w>.$"
Theorem. \( EQ_{TM} \) is neither Turing-recognizable nor co-Turing-recognizable.

Proof. We show \( A_{TM} \leq_m EQ_{TM} \). Why does this help?

\( EQ_{TM} \) is not Turing-recognizable.

Theorem. \( EQ_{TM} \) is neither Turing-recognizable nor co-Turing-recognizable.

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Proof. We show \( A_{TM} \leq_m EQ_{TM} \). Why does this help?
Exercises

1. Show that $A_{TM}$ is not mapping reducible to $E_{TM}$.
   (Hint: Use the fact that $A_{TM}$ is not Turing-recognizable whereas $E_{TM}$ is Turing-recognizable.)

2. Show that if $P$ is Turing-recognizable and $P \leq_m \overline{P}$, then $P$ is decidable.