The Classes P and NP

Polynomial Time

Tractable Problems

<table>
<thead>
<tr>
<th>Size $n$</th>
<th>Time complexity function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n$</td>
</tr>
<tr>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td></td>
<td>$n^3$</td>
</tr>
<tr>
<td></td>
<td>$n^5$</td>
</tr>
<tr>
<td></td>
<td>$2^n$</td>
</tr>
<tr>
<td></td>
<td>$3^n$</td>
</tr>
</tbody>
</table>

Polynomially Equivalent Models
Graph Theory

Definition. \( \text{PATH} = \{ \langle G, s, t \rangle \mid \exists \text{ a directed path from } s \text{ to } t \text{ in } G \} \)

PATH \( \in \mathbb{P} \)

\( M = \) "On input \( \langle G, s, t \rangle \):
1. Place a mark on node \( s \).
2. Repeat until no additional nodes are marked.
3. Scan all edges of \( G \). If \( (a, b) \) found from marked node to unmarked node, mark \( b \).
4. If \( t \) is marked, accept. Otherwise, reject."

Hamiltonian Paths

Definition. \( \text{HAMPATH} = \{ \langle G, s, t \rangle \mid \exists \text{ a Hamiltonian path from } s \text{ to } t \} \)

Checking for Hamiltonian Paths

\( E = \) "On input \( \langle G, s, t \rangle \):
1. Generate all orderings, \( p_1, p_2, \ldots, p_m \) of the nodes in \( G \).
2. Check whether \( s = p_1 \) and \( t = p_m \).
3. For each \( i = 1 \) to \( n-1 \), check whether \( (p_i, p_{i+1}) \) is an edge in \( G \). If any are not, reject. Otherwise, accept."
Guessing a Solution

\[ N = \text{"On input } \langle G, s, t \rangle:\]
1. Guess an ordering, \( p_1, p_2, \ldots, p_n \), of the nodes in \( G \).
2. Check whether \( s = p_1 \) and \( t = p_n \).
3. For each \( i = 1 \) to \( n-1 \), check whether \((p_i, p_{i+1})\) is an edge in \( G \). If any are not, reject. Otherwise, accept.

The Class NP

\textbf{Definition.} \ NTIME(\( t(n) \)) = \{ L \mid L \text{ is decided in } O(t(n)) \text{ time by an NTM} \}.

\textbf{Corollary.} \ NP = \bigcup_k \ NTIME(n^k).

Nondeterministic Time Complexity

\textbf{Definition.} \ Let \( N \) be a NTM. The \textit{running time} of \( N \) is a function \( f : \mathbb{N} \to \mathbb{N} \), where \( f(n) \) is the maximum number of steps that \( N \) uses on any branch of its computation on any input of length \( n \).

Certificates

NP is the class of languages that have polynomial time verifiers.

What is a certificate for \textit{PATH}?
What is a certificate for \textit{HAMPATH}?
What is a certificate for \textit{COMPOSITE}?
What is a certificate for \textit{HAMPATH}?
The Classes P and NP

P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq \text{EXPTIME}

Proper containment

P = NP?

Exercises

Let CONNECTED = \{ \langle G \rangle \mid G \text{ is a connected undirected graph} \}.
Is CONNECTED in NP? Is CONNECTED in P?

A triangle in an undirected graph is a 3-clique.
Let TRIANGLE = \{ \langle G \rangle \mid G \text{ contains a triangle} \}.
Is TRIANGLE in NP? Is TRIANGLE in P?

Call the graphs G and H isomorphic if the nodes of G may be reordered so that it is identical to H.
Let ISO = \{ \langle G, H \rangle \mid G \text{ and } H \text{ are isomorphic graphs} \}.
Is ISO in NP? Is ISO in P?