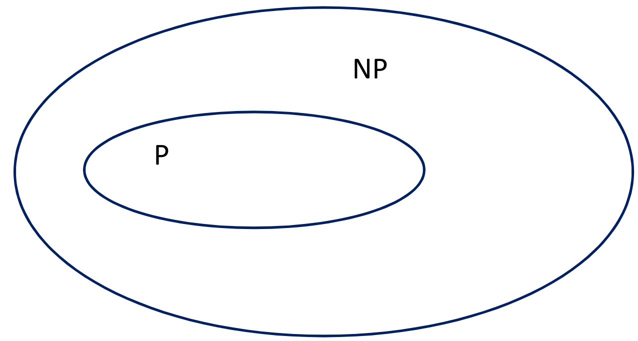
The Hardest Problem In The World

Sipser: Section 7.4 pages 299 – 311

The Classes P and NP?



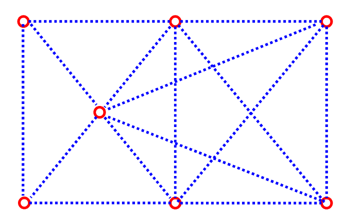
A Famous NP Problem

CNF satisfiability (*CNFSat*): given a Boolean formula *B* in conjunctive normal form (CNF), is there a truth assignment that satisfies *B*?

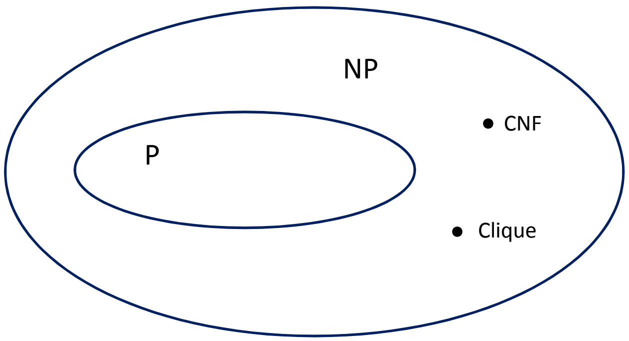
$$(x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2}) \land (x_1 \lor \overline{x_2})$$

A Graph Theory NP Problem

CLIQUE: given a graph $G = (V, E)$ and an integer k, does G contain C_k as a subgraph?

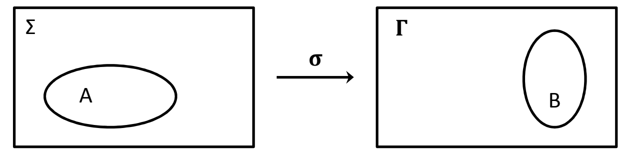


Which Problem is Harder?



Polynomial Time Reduction

Definition. Let $A \subseteq \Sigma$ and $B \subseteq \Gamma$ be decision problems.



We write $A \leq_p B$ and say that *A reduces to B in polynomial time* if there is a polynomial time computable function $\sigma: \Sigma \rightarrow \Gamma$ such that for all problem instances $x \in \Sigma$,

$$x \in A \text{ iff } \sigma(x) \in B.$$

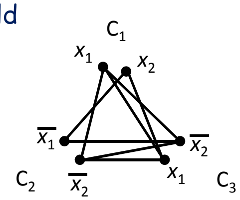
CNF $\leq_p$ Clique

- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

C_1 C_2 C_3

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee \overline{x_2})$$

the construction would yield



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$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

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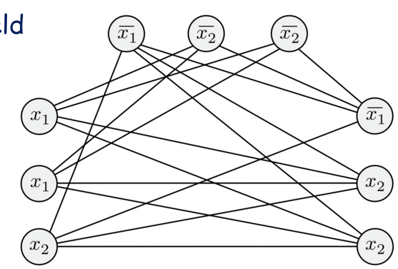
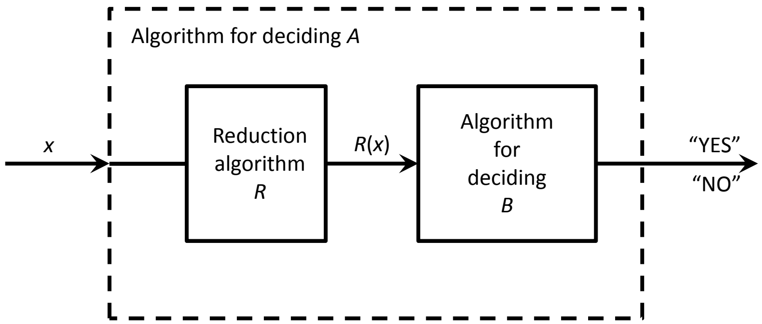


Intuitively, *A* is No Harder than *B*

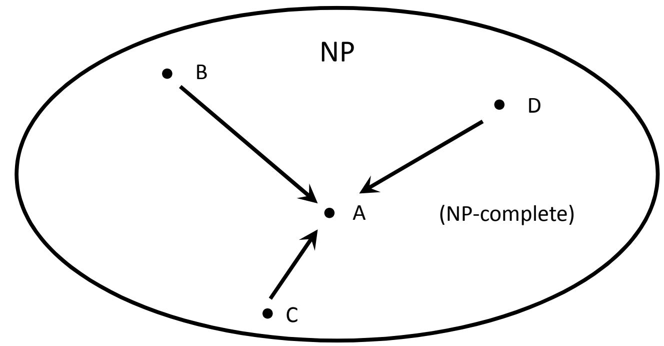
Theorem. If $A \leq_p B$ and B has a polynomial time algorithm, then so does A.

Proof.



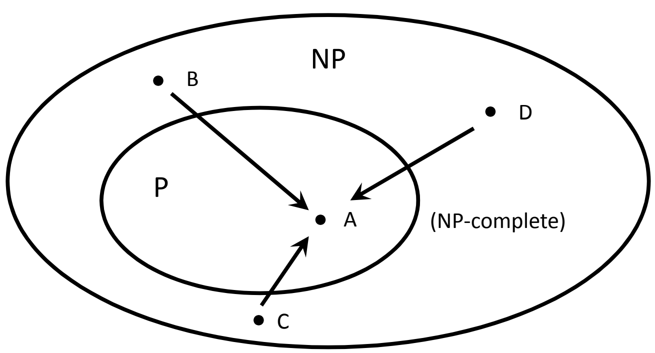
NP's Hardest Problems

Definition. The set $A \in$ NP is *NP-complete* if for all $B \in$ NP, $B \leq_p A$.



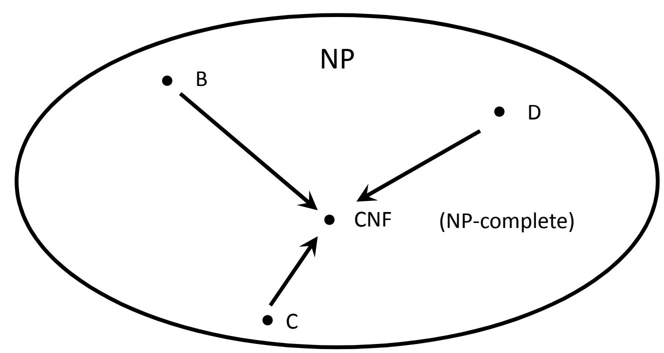
P = NP?

Theorem. If A is NP-complete, then $A \in$ P if and only if P = NP.



Cook's Theorem

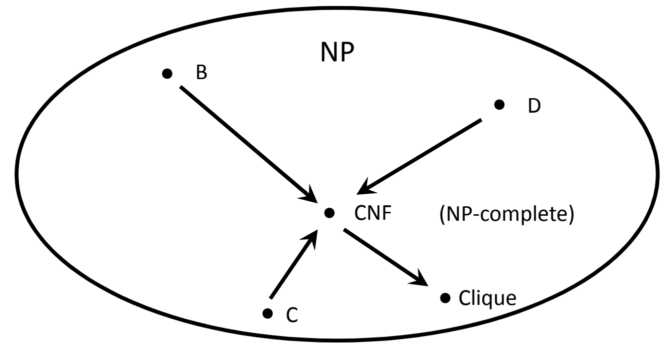
Theorem. If $A \in$ NP then $A \leq_p$ CNFSat.



So What?

- The existence of one "natural" NP-complete problem is an interesting fact for the computer scientist.
- The existence of thousands of "natural" NP-complete problems is an essential fact for the computer scientist.

Clique is NP-Complete



NP-Complete Problems

DOUBLE-SAT = { ‹φ› | φ has at least two satisfying assignments }

Show *DOUBLE-SAT* is in NP. Show *DOUBLE-SAT* is NP-complete. HINT: Reduce *SAT* to *DOUBLE-SAT*. Create a new Boolean formula φ' based on Boolean formula φ such that φ is in *SAT* iff φ' is in *DOUBLE-SAT*.

HALF-CLIQUE = { ‹*G*› | *G* has a clique of size $m/2$ where m is the number of nodes in *G* }

Show *HALF-CLIQUE* is in NP. Show *HALF-CLIQUE* is NP-complete. HINT: Reduce *CLIQUE* to *HALF-CLIQUE*. Create a graph *H* such that ‹*G*,*k*› is in *CLIQUE* iff ‹*H*› is in *HALF-CLIQUE*. Consider the three cases when $k=m/2$, $k > m/2$, and $k < m/2$.