The Hardest Problem In The World

The Classes P and NP?

A Famous NP Problem

CNF satisfiability (CNFSat): given a Boolean formula $B$ in conjunctive normal form (CNF), is there a truth assignment that satisfies $B$?

$$(x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2}) \land (x_1 \lor \overline{x_2})$$

A Graph Theory NP Problem

CLIQUE: given a graph $G = (V, E)$ and an integer $k$, does $G$ contain $C_k$ as a subgraph?
Which Problem is Harder?

Polynomial Time Reduction

**Definition.** Let \( A \subseteq \Sigma \) and \( B \subseteq \Gamma \) be decision problems.

We write \( A \leq_p B \) and say that \( A \) reduces to \( B \) in polynomial time if there is a polynomial time computable function \( \sigma : \Sigma \rightarrow \Gamma \) such that for all problem instances \( x \in \Sigma \),

\[
x \in A \iff \sigma(x) \in B.
\]

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**CNF \leq_p Clique**

- Given a Boolean formula \( B \) in CNF, we show how to construct a graph \( G \) and an integer \( k \) such that \( G \) has a clique of size \( k \) iff \( B \) is satisfiable.

- Given

\[
(x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2}) \land (x_1 \lor x_2)
\]

the construction would yield

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- Given

\[
(x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (x_1 \lor \overline{x_2} \lor x_3)
\]

the construction would yield
Intuitively, $A$ is No Harder than $B$

**Theorem.** If $A \leq_p B$ and $B$ has a polynomial time algorithm, then so does $A$.

**Proof.**

![Diagram of reduction algorithm](image)

Algorithm for deciding $A$  

$R(x)$  

Algorithm for deciding $B$  

"YES"  

"NO"

NP’s Hardest Problems

**Definition.** The set $A \in \text{NP}$ is $\text{NP}$-complete if for all $B \in \text{NP}$, $B \leq_p A$.

![Diagram of NP's hardest problems](image)

P = NP?

**Theorem.** If $A$ is $\text{NP}$-complete, then $A \in \text{P}$ if and only if $\text{P} = \text{NP}$.

![Diagram of P = NP?](image)

Cook’s Theorem

**Theorem.** If $A \in \text{NP}$ then $A \leq_p \text{CNFSat}$.

![Diagram of Cook’s Theorem](image)
So What?

• The existence of one "natural" NP-complete problem is an interesting fact for the computer scientist.

• The existence of thousands of "natural" NP-complete problems is an essential fact for the computer scientist.

Clique is NP-Complete

![Diagram of Clique is NP-Complete]

NP-Complete Problems

**DOUBLE-SAT** = \{ \langle \varphi \rangle \mid \varphi \text{ has at least two satisfying assignments} \}

Show **DOUBLE-SAT** is in NP. Show **DOUBLE-SAT** is NP-complete. HINT: Reduce **SAT** to **DOUBLE-SAT**. Create a new Boolean formula \( \varphi' \) based on Boolean formula \( \varphi \) such that \( \varphi \) is in **SAT** iff \( \varphi' \) is in **DOUBLE-SAT**.

**HALF-CLIQUE** = \{ \langle G \rangle \mid G \text{ has a clique of size } m/2 \text{ where } m \text{ is the number of nodes in } G \}

Show **HALF-CLIQUE** is in NP. Show **HALF-CLIQUE** is NP-complete. HINT: Reduce **CLIQUE** to **HALF-CLIQUE**. Create a graph \( H \) such that \( \langle G,k \rangle \) is in **CLIQUE** iff \( \langle H \rangle \) is in **HALF-CLIQUE**. Consider the three cases when \( k = m/2 \), \( k > m/2 \), and \( k < m/2 \).