The Equivalence of NFAs and DFAs

Sipser: Section 1.2 pages 54 - 58

Theorem. A language is regular if and only if it is accepted by a nondeterministic finite automaton.

Proof. (⇒) Let A be a regular language ...

A finite automaton is a 5-tuple (Q, Σ, δ, q₀, F), where
1. Q is a finite set called the states,
2. Σ is a finite set called the alphabet,
3. δ: Q × Σ → Q is the transition function,
4. q₀ ∈ Q is the start state, and
5. F ⊆ Q is the set of accept states.

A nondeterministic finite automaton is a 5-tuple (Q, Σ, δ, q₀, F), where
1. Q is a finite set called the states,
2. Σ is a finite set called the alphabet,
3. δ: Q × Σ → P(Q) is the transition function,
4. q₀ ∈ Q is the start state, and
5. F ⊆ Q is the set of accept states.
Going the Other Way

**Theorem.** A language is regular if and only if it is accepted by a nondeterministic finite automaton.

**Proof.** \((\Leftarrow\Rightarrow)\) Suppose \(A\) is accepted by a nondeterministic finite automaton ...

Equivalent Languages

**Definition.** Two machines are equivalent if they recognize the same language.

**Theorem.** Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

**Proof.**

You Only Go Around Once

Input 010110

A Simpler Example
Removing Choice

**Proof.**
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA recognizing language $A$.

We construct a DFA $M$ recognizing $A$.

1. $Q' = \mathcal{P}(Q)$.
2. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$.
3. $q_0' = \{q_0\}$.
4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.

What About $\varepsilon$ Arrows?

**Proof.**
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA recognizing language $A$.

We construct a DFA $M$ recognizing $A$.

1. $Q' = \mathcal{P}(Q)$.
2. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$, where $E(R) = \{q \mid q \text{ can be reached from } R \text{ along } 0 \text{ or more } \varepsilon \text{ arrows } \}$.
3. $q_0' = E(\{q_0\})$.
4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.

Modifying Our Construction

**Proof.**
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA recognizing language $A$.

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Equivalent DFA?

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We construct a DFA $M$ recognizing $A$.

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3. $q_0' = E(\{q_0\})$.
4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$. 