Closure Under Regular Operations (using NFAs)
Regular Languages

- What is the definition of regular languages?
- What is the relationship between DFA and NFA?
- What can we conclude from the above information?

Corollary.
Regular Languages

- What is the definition of regular languages?
  - Languages that can be recognized by some DFA.

- What is the relationship between DFA and NFA?
  - Lemma: Every NFA has an equivalent DFA.

- What can we conclude from the above information?

**Corollary.** A language is regular if and only if some NFA recognizes it.
Recall Closure Properties

- Consider some operation OP and a language A:
  - If A is a regular language, then OP(A) is also a regular language.
  - Same as “Regular languages are closed under OP”
Recall Closure Properties

- Consider some operation $OP$ and a language $A$:
  - If $A$ is a regular language, then $OP(A)$ is also a regular language.
  - Same as “Regular languages are closed under $OP$”

- We have proved that regular languages are closed under
  - Complement
  - Union
  - Intersection
  - Which one are we missing?
Recall Closure Properties

• Consider some operation OP and a language A:
  ○ If A is a regular language, then OP(A) is also a regular language.
  ○ Same as “Regular languages are closed under OP”

• We have proved that regular languages are closed under
  ○ Complement
  ○ Union
  ○ Intersection
  ○ Which one are we missing?

• Good news: NFAs can help us out!
Recall Example from Before

$L = \{ w \mid w \text{ is a string of 0s and 1s that starts with 010 or ends with 110}\}.$
Warm Up: Closure Under Unions Using NFAs

**Theorem.** The class of regular languages is closed under the union operation.

**Proof.**
Let’s Be Precise

Theorem. The class of regular languages is closed under the union operation.

Proof. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $L_1$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $L_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $L_1 \cup L_2$. 

Let’s Be Precise

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Construct \( N = (Q, \Sigma, \delta, q_0, F) \) to recognize \( L_1 \cup L_2 \).

1. \( Q = \{q_0\} \cup Q_1 \cup Q_2 \), for a new state \( q_0 \).
2. \( q_0 \) is the start state of \( N \).
3. \( F = F_1 \cup F_2 \).
4. For any \( q \in Q \) and any \( a \in \Sigma \),
Let’s Be Precise

Theorem. The class of regular languages is closed under the union operation.

Proof. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $L_1$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $L_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $L_1 \cup L_2$.
1. $Q = \{q_0\} \cup Q_1 \cup Q_2$, for a new state $q_0$.
2. $q_0$ is the start state of $N$.
3. $F = F_1 \cup F_2$.
4. For any $q \in Q$ and any $a \in \Sigma_e$,
   $$
   \delta(q, a) = \begin{cases} 
   \delta_1(q, a) & \text{if } q \in Q_1 \\
   \delta_2(q, a) & \text{if } q \in Q_2 \\
   \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\
   \emptyset & \text{if } q = q_0 \text{ and } a \neq \varepsilon
   \end{cases}
   $$
Closure of Regular Languages Under Concatenation

Recall definition:

Let $A$ and $B$ be languages. We define the concatenation of $A$ and $B$ as

$$A \circ B = \{ xy \mid x \in A \text{ and } y \in B \}.$$
Closure of Regular Languages Under Concatenation

**Theorem.** The class of regular languages is closed under the concatenation operation.

**Proof.**
Closure of Regular Languages Under Concatenation

**Theorem.** The class of regular languages is closed under the concatenation operation.

**Proof.** Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $L_1$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $L_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $L_1 \circ L_2$

1. $Q = Q_1 \cup Q_2$.
2. $q_0 = q_1$.
3. $F = F_2$.
4. For any $q \in Q$ and $a \in \Sigma$, 

\[
\delta(q, a) = \begin{cases} 
\delta_1(q, a) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\
\delta_1(q, a) & \text{if } q \in F_1 \text{ and } a \neq \varepsilon \\
\delta_1(q, a) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \varepsilon \\
\delta_2(q, a) & \text{if } q \in Q_2 
\end{cases}
\]
Kleene Star

Let $A$ be a language. We define $A^* = \{ x_1x_2\ldots x_k \mid k \geq 0 \text{ and each } x_i \in A \}$. 

Let $A = \{0, 1\}$, let $B = \{ w \mid w \text{ is a string of 0s and 1s containing an even number of 1s } \}$, and let $C = \{ w \mid w \text{ is a string of 0s and 1s containing an odd number of 1s } \}$.

What are $A^*$, $B^*$, and $C^*$?

$A^*$ = All string of 0s and 1s  
$B^*$ = $B$  
$C^*$ = $A^*$ - \{strings of only 0s\}
Closure of Regular Languages Under Kleene Star

Theorem. The class of regular languages is closed under the Kleene star operation.

Proof.
Closure of Regular Languages Under Kleene Star

**Theorem.** The class of regular languages is closed under the concatenation operation.

**Proof.** Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $L$

Construct $N = (Q, \Sigma, \delta, q_0, F')$ to recognize $L^*$

1. $Q = \{q_0\} \cup Q_1$.
2. $q_0$ is the start state of $N$.
3. $F = \{q_0\} \cup F_1$.
4. For any $q \in Q$ and $a \in \Sigma$, $
   \delta(q, a) = \begin{cases} 
   \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\
   \delta_1(q, a) & \text{if } q \in F_1 \text{ and } a \neq \epsilon \\
   \delta_1(q, a) \cup \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon \\
   \{q_1\} & \text{if } q = q_0 \text{ and } a \neq \epsilon \\
   \emptyset & \text{ otherwise}
   \end{cases}$
NFA with a Single Accept State

**Theorem.** Prove that any NFA can be converted to an equivalent one that has a single accept state.

**Proof.**