Projection Math
Computer Graphics
Scott D. Anderson

1 Perspective Projection for Pinhole Camera

We want to compute a point (y_p, z_p) that is the projection of a point in the scene at (y, z). We've eliminated the x coordinate by slicing through the scene and camera at $x = 0$.

First, notice that $z_p = -d$, because the depth of the projected point must fall on the projection plane at the back of the camera, which is at depth d. Therefore, by similar triangles:

$$\frac{y_p}{-d} = \frac{y}{z}$$

so,

$$y_p = \frac{-yd}{z} = \frac{-y}{z/d}$$

All the quantities on the right are known: the point that is being projected, and the depth of the camera.

2 Perspective for Synthetic Camera

For the synthetic camera, we just put the projection plane on the other side of the origin. Again, by similar triangles:

$$y_p = \frac{yd}{z} = \frac{y}{z/d}$$

3 Homogeneous Coordinates Redux

We can represent a point in homogeneous coordinates without forcing $w = 1$. Instead, we can allow all the components to be scaled by w. In other words:

$$P = \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

We can recover the "real" point by dividing all the components by w, which make the last component 1 again.

4 Perspective Division

We can use the idea in the previous section to let us to the perspective computation using matrix multiplication. Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

Suppose we start with a point $P = (x, y, z, 1)$ in homogeneous coordinates. We can find the projection of P by multiplying by M:

$$Q = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

or, we recover the "real" point as:

$$Q = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ \frac{z}{z/d} \end{bmatrix}$$

So Q is the projection of P! M is the perspective projection matrix.

5 Field of View (Y)

How would we compute the angle for the field of view y, which we'll call θ given a particular pinhole camera? We can draw a right triangle where the base is of length d (the depth of the pinhole camera) and the height is half of h — the height of the image plane. Given that, we can compute.

$$\theta = 2\tan^{-1}\left(\frac{h/2}{d}\right)$$