

Vector Math

Computer Graphics

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1 Dot Product

The notation $v \cdot w$ means the *dot product* or *scalar product* or *inner product* of two vectors, v and w . In abstract mathematics, we can talk about choosing an inner product on a vector space, and that inner product has to satisfy certain properties:

- $v \cdot w \in \mathfrak{R}$ That is, the dot product of two vectors is just a real number, a scalar (not a vector).
- $\forall v, v \cdot v = 0$ iff $v = \vec{0}$. That is, the dot product of a vector with itself is zero if and only if the vector is the zero vector. In 3D, the zero vector is $(0,0,0)$.
- (commutativity) $v \cdot w = w \cdot v$
- (linearity) $v \cdot (au + bw) = a(v \cdot u) + b(v \cdot w)$. That is, the dot product of a vector v with the linear combination of two other vectors, u and w , is the same as the linear combination of the two dot products.

But in this course, we're going to be very concrete.

$$v \cdot w = \sum_{i=0}^n v_i w_i$$

In fact, since we only are using 3 dimensions:

$$v \cdot w = v_0 w_0 + v_1 w_1 + v_2 w_2$$

You can easily check that this definition of a dot product satisfies the required properties.

Note that calculating one element of a matrix multiplication is the same thing as a dot product: the result element is the dot product of a row from the first matrix with a column from the second matrix.

1.1 Example

Suppose we have the following two vectors:

$$\begin{aligned} v &= (1, 2, 3) \\ w &= (6, 5, 4) \end{aligned}$$

The dot product of those vectors is:

$$v \cdot w = 1 * 6 + 2 * 5 + 3 * 4 = 6 + 10 + 12 = 28$$

Note that the dot product of two vectors is a *scalar*: a real number, not a vector.

2 Length

Given the dot product, we can define the length of a vector:

$$|v| = \sqrt{v \cdot v}$$

This is just our old friend the Pythagorean theorem in disguise! Why? Because the dot product of a vector with itself ends up squaring each element and adding them up.

Note that we can rescale (shrink or grow) any vector (except the zero vector) to have unit length. This is called *normalizing*.

$$w = \frac{v}{|v|}$$

$$|w| = 1$$

2.1 Example

What is the length of the vector $v = (2, 3, 6)$?

$$|v| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

If we *normalize* v , what do we get? Let $w = v/|v|$, we get

$$w = (2, 3, 6)/7 = (2/7, 3/7, 6/7) \approx (0.28, 0.43, 0.86)$$

You can easily check that $|w| = 1$. The vector w is in exactly the same *direction* as v , but it's one unit long.

3 Angles Between Vectors

The angle between two vectors v and w can be found using the following formula. Amazing, but true.

$$\cos(\theta) = \frac{v \cdot w}{|v||w|}$$

which I like to also think of as:

$$\cos(\theta) = \frac{v}{|v|} \cdot \frac{w}{|w|}$$

In other words, it's just the dot product of the two normalized vectors. That is, if both of two vectors v and w are normalized (a length of one unit), the formula for the cosine of the angle between them becomes the following, which is remarkably simple:

$$\cos(\theta) = v \cdot w$$

This is good news, because a dot product is very quick to compute: in 3D, it's just 3 multiplies and 2 adds. It turns out that the cosine of an angle is often desired in computer graphics, so being able to compute it so simply is enormously valuable. In fact, there are occasions when the tail wags the dog: the cosine of the angle is used because it's so quick to compute.

3.1 Example 1

Given the following vectors, what is the cosine of the angle between them?

$$\begin{aligned}v &= (2, 3, 6) \\w &= (9, 6, 2)\end{aligned}$$

Since we already know the length of v , let's first find the length of w :

$$|w| = \sqrt{w \cdot w} = \sqrt{9^2 + 6^2 + 2^2} = \sqrt{81 + 36 + 4} = \sqrt{121} = 11$$

Therefore, the cosine of the angle between v and w is:

$$\cos(\theta) = \frac{v \cdot w}{7 * 11} = \frac{2 * 9 + 3 * 6 + 6 * 2}{77} = \frac{18 + 18 + 12}{77} = \frac{48}{77} \approx 0.62$$

If, for some reason, we needed the actual angle, we could compute the arc cosine, but we'll almost never need that. (This angle is about 51 degrees.)

3.2 Example 2

Suppose we have the vectors $x = (1, 0, 0)$ and $y = (0, 1, 0)$. That is, these are unit vectors that point in the direction of the X and Y axes. It's easy to see that the dot product between these is zero.

What does this mean? If the cosine of an angle is zero, that means the angle is a right angle (ninety degrees). That is, these two vectors are *perpendicular*. (We're not surprised that they're perpendicular, but it's nice to see that the calculation confirms it.) In linear algebra, the term *orthogonal* is used for vectors that are perpendicular. In geek-speak, we say that two things are "orthogonal" if one doesn't affect the other. This makes sense because we can translate an object by multiples of the vector x without affecting its Y coordinate, and translate it by multiples of the vector y without affecting its X coordinate.

3.3 Example 3

What can we say about the angle, θ between the vectors:

$$\begin{aligned}v &= (2, 3, 6) \\u &= (-6, 10, -3)\end{aligned}$$

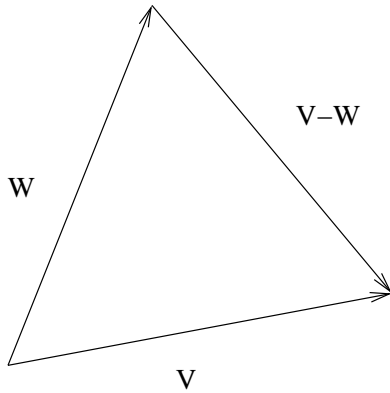
The cosine of the angle is found by:

$$\cos(\theta) = \frac{(2, 3, 6) \cdot (-6, 10, -3)}{|v||u|} = \frac{-12 + 30 - 18}{|v||u|} = \frac{0}{|v||u|} = 0$$

So, these vectors are perpendicular (orthogonal).

4 Applying the Law of Cosines

This section proves that the dot product of normalized vectors is the angle between them. You can skip it if you're not interested.



According to the law of cosines, from trigonometry:

$$|v - w|^2 = |v|^2 + |w|^2 - 2|v||w| \cos(\theta)$$

Where θ is the angle between v and w . With some algebra, we get:

$$\cos(\theta) = \frac{|v|^2 + |w|^2 - |v - w|^2}{-2|v||w|}$$

All the terms in $v - w$ are of the form

$$(v_0 - w_0)^2$$

multiplying out to

$$v_0^2 - 2v_0w_0 + w_0^2$$

That means that all the squared terms in the numerator disappear, the -2 cancels with the denominator, and we are left only with the cross terms:

$$\cos(\theta) = \frac{\sum_{i=0}^n v_i w_i}{|v||w|}$$

but that numerator is just the dot product of v and w , so

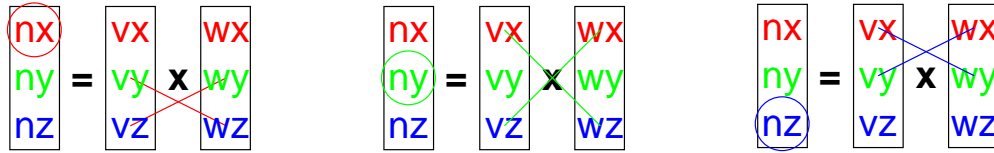
$$\cos(\theta) = \frac{v \cdot w}{|v||w|}$$

5 Cross Product

The dot product of two vectors produces a *scalar*. The cross product produces a *vector*: one that is perpendicular to *both* of them. In 3D, the cross product is:

$$\begin{aligned} \vec{n} &= \vec{v} \times \vec{w} \\ &= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \times \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \\ &= \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix} \end{aligned}$$

Note that order matters. $v \times w = -w \times v$. Notice also that in computing n_x , you use only terms from the Y and Z components, and similarly for n_y and n_z . There's a nice graphical way to remember the way to compute this. Looking at the following figures, multiply the terms connected by lines, and subtract one line from another. For the X and Z, you subtract the major diagonal (sloping down to the right) from the minor diagonal, and for Y, you subtract them in the opposite order (or just negate the result).



5.1 Direction of the cross product

What direction does the vector go in? For that, we need the *right-hand rule*: point the fingers of your right hand in the direction of the first vector, sweep them towards the second, and your thumb points in the direction of the cross product. Alternatively, point your thumb in the direction of the first vector, your index in the direction of the second, and your middle finger in the direction of the cross-product.

Interestingly, $z = x \times y$, $x = y \times z$ and $y = z \times x$. That is, a normalized vector parallel to the z axis, $(0,0,1)$ is the cross product of vectors for the x axis, $(1,0,0)$, and the y axis, $(0,1,1)$.

5.2 Length of the cross product

The length of the cross-product divided by the lengths of the two vectors gives the sine of the angle between the two vectors:

$$\sin(\theta) = \frac{|v \times w|}{|v||w|}$$

5.3 Example

Find the cross product of the following two vectors:

$$\begin{aligned} v &= (2, 3, 6) \\ w &= (9, 6, 2) \end{aligned}$$

We'll write the vectors vertically in order to find the result.

$$\begin{aligned} \vec{n} &= \vec{v} \times \vec{w} \\ &= \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \times \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 * 2 - 6 * 6 \\ 9 * 6 - 2 * 2 \\ 2 * 6 - 9 * 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 6 - 36 \\ 54 - 4 \\ 12 - 27 \end{bmatrix} \\
&= \begin{bmatrix} -30 \\ 50 \\ -15 \end{bmatrix}
\end{aligned}$$

We already know from previous examples that $|v| = 7$ and $|w| = 11$. The sine of the angle between v and w is

$$\sin(\theta) = \frac{|n|}{|v||w|} = \frac{\sqrt{900 + 2500 + 225}}{77} = \frac{\sqrt{3625}}{77} \approx 0.78$$

Earlier, in section 3.1, we found that the cosine of the angle between v and w is $48/77$. Therefore, we should be able to confirm that:

$$1 = \cos^2(\theta) + \sin^2(\theta) = (48/77)^2 + (\sqrt{3625}/77)^2 = (2304 + 3625)/77^2 = 5929/5929$$

Whew!

5.4 Normal Vectors

A vector that is perpendicular to a surface, such as a plane or a sphere, is said to be *normal* to it. It's therefore called a normal vector. (Other vectors are not *abnormal*.)

Normal vectors for surfaces are crucial in lighting calculations, because the normal vector tells us the orientation of the surface, and the orientation is important to how light falls on the surface and where it reflects to.

A common way to find a normal vector for plane is to find two vectors that lie in the plane and take their cross product. To find a normal vector for a surface, find a plane that is tangent at the desired point, and find the normal vector for that plane. We'll see examples of this later in the course.