Fractional Weak Discrepancy and Split Semiorders

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Dedicated to Martin C. Golumbic
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ABSTRACT

The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ was introduced in [8] as the minimum nonnegative k for which there exists a function $f : V \to \mathbf{R}$ satisfying (i) if $a \prec b$ then $f(a)+1 \leq f(b)$ and (ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. In this paper we generalize results in [9, 10] on the range of wd_F for semiorders to the larger class of split semiorders. In particular, we prove that for such posets the range is the set of rationals that can be represented as r/s for which $0 \leq s - 1 \leq r < 2s$.

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Class of Posets	Forbidden Subposets
linear order	no $\mathbf{1}+\mathbf{1}$
weak order	no $\mathbf{2}+\mathbf{1}$
semiorder	no $\mathbf{3}+\mathbf{1}$, no $\mathbf{2}+\mathbf{2}$
interval order	no $\mathbf{2}+\mathbf{2}$

Table 1: Classes of posets characterized in terms of forbidden subposets.

1 Introduction

In this paper we will consider irreflexive posets $P = (V, \prec)$, and write $x \parallel y$ when elements x and y in V are incomparable. Of particular importance to us will be the posets $\mathbf{r}+\mathbf{s}$ consisting of two disjoint chains, one with r elements and one with s elements, where $x \parallel y$ whenever x and y are in different chains. The order $\mathbf{3}+\mathbf{1}$ is shown in Figure 1.

We focus on the fractional weak discrepancy of split semiorders and begin with some background on this and related classes of orders. For additional background and context we refer the reader to [3] and [4].

1.1 Split semiorders and related classes

The four classes of posets: linear orders, weak orders, semiorders, and interval orders, are important both because they arise in applications and also because they have elegant characterizations. Each of these classes can be characterized in terms of forbidden subposets of the form $\mathbf{r}+\mathbf{s}$ as detailed in Table 1. Note that this implies the following inclusions:

$$\{\text{linear orders}\} \subseteq \{\text{weak orders}\} \subseteq \{\text{semiorders}\} \subseteq \{\text{interval orders}\}.$$

These classes also have alternative definitions in terms of interval representations. Such representations are useful in constructions as well is in proofs by contradiction. A poset $P = (V, \prec)$ is an *interval order* if each element $x \in V$ can be assigned an interval $I(v) = [L(v), R(v)]$ in the real line so that $x \prec y$ precisely when $I(x)$ is completely to the left of $I(y)$, that is $R(x) < L(y)$. A *semiorder* (*unit interval order*) is an interval order with a representation in which each interval has the same length. By appropriate scaling, we may assume each interval has length 1.

Linear orders and weak orders can also be defined in this way where each element is assigned a real number (i.e., a degenerate interval). A poset $P = (V, \prec)$ is a *linear order* if each $v \in V$ can be assigned a distinct real number $f(v)$ so that $x \prec y$ if and only if $f(x) < f(y)$. A weak order is similar except the values $f(v)$ need not be distinct, so incomparabilities may occur. These representational definitions are illustrated in Table 2.

Observe that for the first three classes in Table 1, the forbidden subposets are those $\mathbf{r}+\mathbf{s}$ where $r \geq 1$, $s \geq 1$, and $r+s = M$ for $M = 2, 3, 4$, respectively.

Class of Posets	v assigned $I_v = [L(v), R(v)]$	$x \prec y$ iff
interval order		$R(x) < L(y)$
semiorder	$R(v) = L(v) + 1$	$R(x) < L(y)$
weak order	$f(v) = L(v) = R(v)$	$f(x) < f(y)$
linear order	$f(v) = L(v) = R(v)$, $f(x) \neq f(y)$ for $x \neq y$	$f(x) < f(y)$

Table 2: Classes of posets characterized in terms of representations.

Such orders are called $(M, 2)$*-free* in [13]. More generally, an order is (M, t)*-free* if it contains no poset of the form $\mathbf{r_1} + \mathbf{r_2} + \cdots + \mathbf{r_t}$ where $r_1 + r_2 + \cdots + r_t = M$.

A next natural class to consider is the class of $(5, 2)$-free posets, that is, the posets characterized as having no induced $\mathbf{4+1}$ and no induced $\mathbf{3+2}$. This class is called the *subsemiorders* in [3]. Unfortunately, the class of subsemiorders has no known characterization in terms of representations, thus we instead consider a subclass called split semiorders.

Definition 1 A poset $P = (V, \prec)$ is a *split semiorder* if each $v \in V$ can be assigned an interval $I(v) = [L(v), R(v)]$ of unit length u (with $R(v) = L(v) + u$) and a point $C(v) \in I(v)$ so that $x \prec y$ if and only $C(x) < L(y)$ and $R(x) < C(y)$. The point $C(v)$ is called the *point core* or *splitting point* of the interval $I(v)$ and the representation is called a *unit point-core representation.*

Given a unit point-core representation of a split semiorder, a comparability occurs between elements x and y precisely when neither interval $I(x), I(y)$ contains the other interval's splitting point. In the literature on tolerance graphs, split semiorders are also referred to as unit point-core bitolerance orders [6].

Any representation of a poset by real intervals is said to be *unit* if all the intervals in the representation have the same length and *proper* if no interval properly contains another. Sometimes a proper representation is more convenient to construct than a unit representation and thus the following remark can be helpful. Its proof follows from Theorem 10.3 of [6].

Remark 2 A poset P is a split semiorder if and only if it satisfies Definition 1 with a proper representation by intervals $I(v)$ and splitting points $C(v)$ rather than a unit representation.

Every semiorder P has a unit point-core representation obtained by supplementing any unit interval representation P with a point-core assignment C such that $C(v) = L(v)$ for all $v \in V$. Thus, *every semiorder is a split semiorder.* However the containment is proper since $\mathbf{3+1}$ is a split semiorder that is not a semiorder (see Figure 1). The posets $\mathbf{4+1}$ and $\mathbf{3+2}$ are not split semiorders. The details of these proofs appear in [4] and also in Chapter 10 of [6]. Thus split semiorders are indeed $(5, 2)$-free.

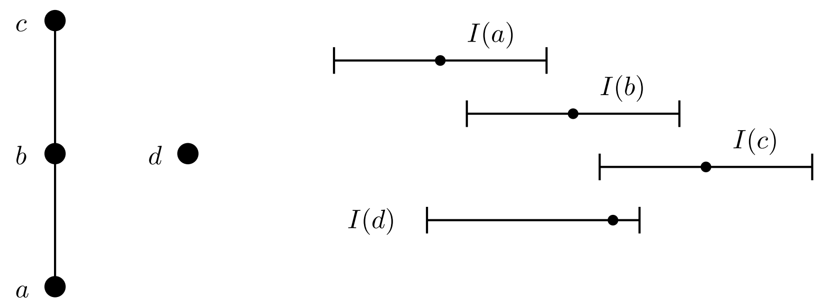

Figure 1: The order $\mathbf{3}+\mathbf{1}$ and a representation of it as a split semiorder.

1.2 Fractional Weak Discrepancy

A poset $P = (V, \prec)$ is a *weak order* if there exists a real-valued function $f : V \to \mathbf{R}$ so that $x \prec y$ if and only if $f(x) < f(y)$. We can think of such a function as assigning a rank to each element of P in such a way that respects the ordering $\prec$ and gives incomparable elements equal rank. Sometimes it is desirable to rank the elements of a poset that is not a weak order. For example, a poset could represent a set V of employees partially ordered by their value to a company and the function value $f(v)$ could represent employee v's salary. We want such a ranking function to satisfy two "fairness" conditions: first that a more valuable employee receives a significantly higher salary, and second that we seek to minimize the largest discrepancy in salaries between incomparable employees. These conditions are made more formal in the following definition.

Definition 3 The *fractional weak discrepancy* $wd_F(P)$ of a poset $P = (V, \prec)$ is the minimum nonnegative real number k for which there exists a function $f : V \to \mathbf{R}$ satisfying
(i) if $a \prec b$ then $f(a) + 1 \leq f(b)$ ("up" constraints)
(ii) if $a \parallel b$ then $|f(a) - f(b)| \leq k$. ("side" constraints)
Such a function is called an *optimal labeling* of P.

To illustrate this definition, Figure 2 shows a poset Z with a labeling function that satisfies conditions (i) and (ii) for $k = 4/3$, thus $wd_F(Z) \leq 4/3$. We will show later that this is indeed an optimal labeling and thus $wd_F(Z) = 4/3$.

Fractional weak discrepancy was first defined in [8] and studied further in [9, 10, 11]. The integer version of the problem (where each function value $f(v)$ must be an integer) was introduced in [13] as the *weakness* of a poset, and studied further as *weak discrepancy* in [5, 12]. The poset $\mathbf{3}+\mathbf{1}$ shown in Figure 1 has weak discrepancy and fractional weak discrepancy equal to 1

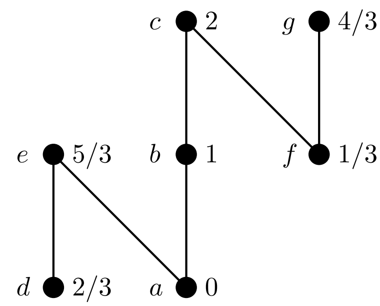

Figure 2: The poset Z with a labeling satisfying the conditions of Definition 3 with $k = 4/3$.

with the following optimal labeling: $f(a) = 0$, $f(b) = 1$, $f(c) = 2$, $f(d) = 1$. Furthermore, any poset P containing an induced $\mathbf{3}+\mathbf{1}$ will have $wd_F(P) \geq 1$.

The existence of a labeling of a poset P satisfying conditions (i) and (ii) of Definition 3 for a particular k shows that $wd_F(P) \leq k$. We seek a certificate to demonstrate that a labeling is optimal, that is, a structure that ensures $wd_F(P) \geq k$. The presence of an $\mathbf{r}+\mathbf{s}$ in a poset P gives a substructure whose elements we can traverse by traveling up one chain, then to the bottom of the second chain, then up the second chain, and then back to the bottom of the first chain, all the while respecting the ordering in P. This is generalized in the following definition.

Definition 4 A *forcing cycle* C of poset $P = (V, \prec)$ is a sequence $C : x_0, x_1, \ldots, x_m = x_0$ of $m \geq 2$ elements of V for which $x_i \prec x_{i+1}$ (an *up step*) or $x_i \parallel x_{i+1}$ (a *side step*) for each $i : 0 \leq i < m$. If C is a forcing cycle, we write $\text{up}(C) = |\{i : x_i \prec x_{i+1}\}|$ and $\text{side}(C) = |\{i : x_i \parallel x_{i+1}\}|$.

In [2], forcing cycles are called *picycles* (preference-indifference cycles). Note that all forcing cycles C have $\text{up}(C) \geq 0$ and $\text{side}(C) \geq 2$.

In our example, $P = \mathbf{3}+\mathbf{1}$ from Figure 1, the following are forcing cycles: $C_1 : a \prec b \prec c \parallel d \parallel a$ with $\text{up}(C_1) = 2$ and $\text{side}(C_1) = 2$, and $C_2 : b \prec c \parallel d \parallel b$ with $\text{up}(C_1) = 1$ and $\text{side}(C_1) = 2$. There are many forcing cycles in the poset Z of Figure 2 including:

$$C_3 : a \prec b \prec c \parallel d \prec e \parallel f \prec g \parallel a$$

with $\text{up}(C_3) = 4$ and $\text{side}(C_3) = 3$.

Once a starting point is specified, a forcing cycle can be described as p alternating sequences U_j of u_j consecutive up steps and S_j of s_j consecutive side steps for $j = 1, 2, \ldots, p$. Thus $\text{up}(C) = \sum_1^p u_j$ and $\text{side}(C) = \sum_1^p s_j$. For example, C_2 above has $p = 1$ with $u_1 = 1$ and $s_1 = 2$, while C_3 has $p = 3$ with $u_1 = 2$, $s_1 = 1$, $u_2 = 1$, $s_2 = 1$, $u_3 = 1$, and $s_3 = 1$. This notation will be useful in Section 2.

Forcing cycles provide the main tool for proving results about fractional weak discrepancy. The ratio $\frac{\mathrm{up}(C)}{\mathrm{side}(C)}$ of a forcing cycle C in poset P provides a lower bound for the fractional weak discrepancy of P, as detailed the following theorem.

Theorem 5 ([8]) Let $P = (V, \prec)$ be a poset with at least one incomparable pair. Then $wd_F(P) = \max_C \frac{\mathrm{up}(C)}{\mathrm{side}(C)}$, where the maximum is taken over all forcing cycles C in P.

In our example $P = \mathbf{3} + \mathbf{1}$, forcing cycle C_1 achieves the maximum ratio and we get $wd_F(P) = 2/2 = 1$. In the example Z in Figure 2, the labeling shows $wd_F(Z) \leq 4/3$ and the forcing cycle C_3 shows $wd_F(Z) \geq 4/3$ thus $wd_F(Z) = 4/3$.

Theorem 5 implies that the fractional weak discrepancy of any poset will be a rational number, but which rational numbers are actually achieved? In this paper we fully answer this question for split semiorders.

2 An upper bound for wd_F of a split semiorder

In this section we give an upper bound for the fractional weak discrepancy of a split semiorder. In [9] we proved that $wd_F(P) < 1$ if and only if P is a semiorder. In Theorem 6, we generalize this result to prove that if P is a split semiorder then $wd_F(P) < 2$.

Theorem 6 *Let P be a split semiorder and C be a forcing cycle in P. Then $\mathrm{up}(C) \leq 2(\mathrm{side}(C) - 1)$. In particular, $wd_F(P) < 2$.*

It suffices to prove $\mathrm{up}(C) \leq 2(\mathrm{side}(C) - 1)$ since then by applying Theorem 5 to an *optimal* forcing cycle C we find $wd_F(P) = \frac{\mathrm{up}(C)}{\mathrm{side}(C)} \leq 2\left(1 - \frac{1}{\mathrm{side}(C)}\right) < 2$. We will see by results in Section 3 that both upper bounds in Theorem 6 are the best possible ones for split semiorders.

The rest of this section is devoted to proving that $\mathrm{up}(C) \leq 2(\mathrm{side}(C) - 1)$ for an arbitrary forcing cycle in a split semiorder. We will assume an instance where it is false for some C. We then apply an algorithm that moves along the cycle through successive sequences of up steps and of side steps and builds a stack K of elements of C. Finally we derive a contradiction from K, thus completing the proof of the theorem.

2.1 The algorithm

Throughout the remainder of Section 2 we will make the following *background assumptions* for the algorithm:

$$\begin{aligned}
&P \text{ is a split semiorder with a fixed unit point-core representation,}\\
&C \text{ is a forcing cycle in } P, \qquad (1)\\
&r = \mathrm{up}(C),\ s = \mathrm{side}(C), \text{ and } r > 2(s-1).
\end{aligned}$$

The algorithm consists of the following three stages. We prove the assertions of each stage in Sections 2.2 and 2.3.

1. Preprocessing: Let C consist of p alternating sequences of u_j consecutive up steps and s_j consecutive side steps, $j = 1, 2, \ldots, p$. If necessary, relabel C to start the cycle at the beginning of a sequence of up steps and so that the partial sums of $\Sigma_{j=1}^{p}(u_j - \lambda_j)$ are nonnegative, where

$$\lambda_j = \begin{cases} 2s_j - 1, & \text{if } j = p \\ 2s_j, & \text{otherwise.} \end{cases} \qquad (2)$$

2. Initialization: (step 0) Place the first element of C on the stack K.

3. Iteration: For each $j = 1, 2, \ldots, p$,

   (jth up step) Add the next u_j elements of C, corresponding to the next sequence U_j of up steps, to the top of K.

   (jth side step) Remove the top λ_j elements from K.

We iterate these steps until we return to the beginning of C and prove that the stack never empties during the iteration. We use the unit point-core representation of P to prove that after each step of the algorithm, the order of elements on the stack K respects the partial order of P. We then use the structure of K to show that C is not a forcing cycle, a contradiction.

2.2 Preprocessing to obtain a good starting point

We have taken a forcing cycle C with $r > 2(s-1)$. We may choose to start the cycle at an element x_0 that is the beginning of a sequence of up steps, i.e., if C contains m elements then $x_{m-1} \parallel x_m = x_0 \prec x_1$. We call x_0 an *upward starting point* for C.

Our goal in preprocessing is to find an upward starting point for C so that the stack K we build during the iteration never empties. We will accomplish this by finding an upward starting point for which the partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are all nonnegative. This is done in Proposition 9, which we prove after the following lemmas.

Lemma 7 *Under the background assumptions in* (1), $\sum_{j=1}^{p}(u_j - \lambda_j) \geq 0$.

Proof. Since $r > 2(s-1)$ and $s = \sum_{j=1}^{p} s_j$, equation (2) implies

$$\begin{aligned} \sum_{j=1}^{p}(u_j - \lambda_j) &= \sum_{j=1}^{p} u_j - \sum_{j=1}^{p} \lambda_j = r - \sum_{j=1}^{p} 2s_j + 1 \\ &> 2(s-1) - 2s + 1 \\ &= -1. \end{aligned}$$

Since both sides are integers the result follows. □

We will also need the fact that whenever the sum of a finite number of real numbers is nonnegative, there is a cyclic permutation of the terms that makes all the partial sums nonnegative. This result is an immediate consequence of Lemma 8 in [11].

Lemma 8 *Let* $\tau = \tau_p : t_1, t_2, \ldots, t_p$ *be a finite sequence of* integers *with* $\sum_{j=1}^{p} t_j \geq 0$*. There exists an index* q *with* $1 \leq q \leq p$ *so that the partial sums of the sequence* $\tau_q : t_{q+1}, t_{q+2}, \ldots, t_p, t_1, t_2, \ldots, t_q$ *are all nonnegative.*

Proposition 9 *Under the background assumptions in* (1)*, there is an upward starting point for* C *for which the partial sums of* $\sum_{j=1}^{p}(u_j - \lambda_j)$ *are all nonnegative.*

Proof. We determine q by applying Lemma 8 to the sequence $\{u_j - \lambda_j\}$ and then let the starting point of C be

$$x_{u_1+s_1+\cdots+u_q+s_q},$$

the element that completes the first q alternating sequences of up and side steps. We then relabel the elements of C so that x_0 again denotes the starting point. It is easy to check that x_0 is now an upward starting point for C for which the partial sums of $\sum_{j=1}^{p}(u_j - \lambda_j)$ are all nonnegative. □

This completes the justification of the prepocessing step of our algorithm.

2.3 Initialization and iteration

We initialize the stack K with the upward starting point x_0 and then iteratively add the next sequence of u_j elements to K and remove λ_j elements from it, for $j = 1, 2, \ldots, p$.

We will use the following notation to help describe the evolution of the stack K during the algorithm. This is summarized in Table 3 along with other notation from this section. Let β_j be the first element added to the stack during the jth up step and let α_j be the top element of the stack after the jth side step. Denote the elements on the stack after the jth up step, from the top of the stack down, by $b_1, b_2, \ldots$. Then $b_{u_j} = \beta_j$ and the top u_j elements of K correspond to the j^{th} sequence of up steps in C, namely $U_j : \beta_j = b_{u_j} \prec \cdots \prec b_2 \prec b_1$.

In the forcing cycle C, U_j is followed by s_j elements corresponding to the next sequence of side steps, $S_j : d_1 \parallel d_2 \parallel \cdots \parallel d_{s_j}$. At the ends of this sequence we have

$$b_1 \parallel d_1 \quad \text{for} \quad 1 \leq j \leq p, \tag{3}$$

$$d_{s_j} \prec \beta_{j+1} \quad \text{for} \quad 1 \leq j \leq p-1. \tag{4}$$

Proposition 10 *Under the background assumptions in* (1)*, the stack* K *never empties during the algorithm.*

Proof. The number of elements on the stack after the jth up step of the algorithm is $1 + \sum_{l=1}^{j-1}(u_l - \lambda_l) + u_j$. The number after the succeeding jth side step is $1 + \sum_{l=1}^{j}(u_l - \lambda_l)$. By Proposition 9, there are always at least two elements on the stack after the jth up step and at least one after the jth side step. Thus the stack never empties during the algorithm. □

Before continuing, it will be useful to note the following lemma.

Lemma 11 *Let P be a split semiorder with a unit point-core representation and let $v \parallel w$ in P.*

(a) $L(v) \leq R(w)$ and $L(w) \leq R(v)$.

(b) If $t \prec u \prec v \parallel w$ in P, then $R(t) < R(w)$ and $C(t) < C(w)$.

(c) If $t \prec u \prec v \parallel w \prec x$, then $t \prec x$.

Proof. Since $w \not\prec v$, by Definition 1 either (i) $C(v) \leq R(w)$ or (ii) $L(v) \leq C(w)$.

(a) In case (i) we have $L(v) \leq C(v) \leq R(w)$. In case (ii) we have $L(v) \leq C(w) \leq R(w)$. So $L(v) \leq R(w)$ is true in both cases and $L(w) \leq R(v)$ follows by symmetry.

(b) In case (i) we have $R(t) < C(v) \leq R(w)$. Similarly, $R(u) < C(v) \leq R(w)$ and since this is a unit representation, $L(u) < L(w)$. Thus

$$C(t) < L(u) < L(w) \leq C(w).$$

In case (ii), again by Definition 1, we have

$$C(t) \leq R(t) < C(u) < L(v) \leq C(w) \leq R(w).$$

So in both cases $R(t) < R(w)$ and $C(t) < C(w)$.

(c) Now suppose we also have $w \prec x$. If $x \prec t$ then $w \prec v$, which contradicts $v \parallel w$. If $t \parallel x$ then it is straightforward to check that the chains $t \prec u \prec v$ and $w \prec x$ form a $\mathbf{3}+\mathbf{2}$ in P. Since P is a split semiorder it is $(5,2)$-free, so this is a contradiction. Thus $t \prec x$. □

Proposition 12 *Under the background assumptions in* (1)*, the order of elements on the stack respects the partial order in P after the jth up step and the jth side step for $j = 1, 2, \ldots, p$.*

Proof. Since we only remove elements from the stack during the jth side step, it suffices to prove the result only for the jth up step. We will do this by induction on j.

For the case $j = 1$, the result is true since x_0 is an upward starting point for C. Now suppose the result is true for $1, 2, \ldots, j$, where $1 \leq j \leq p-1$, and prove it is true for $j+1$. We consider the $(j+1)$st up step. We need to prove that $\alpha_j \prec \beta_{j+1}$.

In the preceding (jth) side step we removed the top $\lambda_j = 2s_j$ elements from the stack K, so that for $1 \leq j \leq p-1$ we have

$$\alpha_j = b_{2s_j+1}. \tag{5}$$

j^{th} sequence of up steps in C	$U_j : b_{u_j} \prec \cdots \prec b_2 \prec b_1$
j^{th} sequence of side steps in C	$S_j : d_1 \parallel d_2 \parallel \cdots \parallel d_{s_j}$
After sequence U_j is processed the top of stack K is	$\beta_j = b_{u_j} \prec \cdots \prec b_2 \prec b_1$
After sequence S_j is processed the top of stack K is	α_j
Definition of e_i	$e_i = b_{2i+1}$
From (5) and the definition of e_i	$\alpha_j = b_{2s_j+1} = e_{s_j}$ for $1 \le j \le p-1$

Table 3: Summary of notation used in Section 2.

We consider the cases $s_j = 1$ and $s_j \ge 2$ separately.

Suppose $s_j = 1$. Then we removed the top two elements b_1 and b_2, leaving b_3 and possibly some additional elements on the stack. Thus $\alpha_j = b_3$ and by the induction assumption for j, $b_3 \prec b_2 \prec b_1$. By (3) and (4), $b_1 \parallel d_1 \prec \beta_{j+1}$. By Lemma 11(c), it follows that $\alpha_j = b_3 \prec \beta_{j+1}$. This completes the proof for step $j+1$ when $s_j = 1$.

Now suppose $s_j \ge 2$. Since $j \le p-1$ and thus $\lambda_j = 2s_j$, we can think of the jth side step as removing s_j pairs of elements, one pair at a time. When we have removed i pairs, let $e_i = b_{2i+1}$ denote the element at the top of the stack at that point.

We will prove by a second induction on i that

$$R(e_i) < R(d_i) \quad \text{and} \quad C(e_i) < C(d_i) \quad \text{for} \quad 1 \le i \le s_j. \tag{6}$$

When $i = 1$, $e_1 = b_3$. By the induction assumption for j we have $b_3 \prec b_2 \prec b_1$ and we know $b_1 \parallel d_1$. So Lemma 11(b) proves that (6) is true for $i = 1$.

Now suppose (6) is true for $1, 2, \ldots, i-1$. We will prove it is true for i. By the induction assumptions for j, we have $b_{2(i-1)+3} \prec b_{2(i-1)+2} \prec b_{2(i-1)+1}$, i.e.,

$$e_i = b_{2i+1} \prec b_{2i} \prec b_{2i-1} = e_{i-1}. \tag{7}$$

Furthermore $d_{i-1} \parallel d_i$. In particular $d_i \not\prec d_{i-1}$ so by Definition 1 either (i) $C(d_{i-1}) \le R(d_i)$ or (ii) $L(d_{i-1}) \le C(d_i)$.

In case (i), by Definition 1 and (7) and (6) for $i-1$,

$$R(e_i) < C(b_{2i}) \le R(b_{2i}) < C(e_{i-1}) < C(d_{i-1}) \le R(d_i).$$

In particular $R(b_{2i}) < R(d_i)$ and, since the representation is unit, we also have $L(b_{2i}) < L(d_i)$. Thus,

$$C(e_i) < L(b_{2i}) < L(d_i) \le C(d_i).$$

In case (ii), again note that (6) for $i-1$ implies $R(e_{i-1}) < R(d_{i-1})$ and therefore $L(e_{i-1}) < L(d_{i-1})$. Thus using Definition 1,

$$C(e_i) < L(e_{i-1}) < L(d_{i-1}) \le C(d_i).$$

Also,

$$R(e_i) < C(b_{2i}) < L(e_{i-1}) \leq C(d_i) \leq R(d_i).$$

In both cases $R(e_i) < R(d_i)$ and $C(e_i) < C(d_i)$, completing the induction on i for $1 \leq i \leq s_j$ and proving (6). To complete the induction on j for $1 \leq j \leq p-1$, it remains to show that $\alpha_j \prec \beta_{j+1}$. Recall from (5) that $e_{s_j} = \alpha_j$ and $d_{s_j} \prec \beta_{j+1}$ by (4). Thus using (6), we have $R(e_{s_j}) < R(d_{s_j}) < C(\beta_{j+1})$ and $C(e_{s_j}) < C(d_{s_j}) < L(\beta_{j+1})$. We conclude that $\alpha_j \prec \beta_{j+1}$ as required.

This shows the order of elements on K respects the partial order of P after each up and side step, completing the proof of Proposition 12. □

Note that in the preceding argument we proved (6) for $1 \leq i \leq s_j$ when $1 \leq j \leq p-1$. In fact the argument is equally valid when $j = p$ provided $1 \leq i \leq s_p - 1$. We will make use of this fact in the proof of Theorem 6.

Proof of Theorem 6.

Let C be the forcing cycle $x_0, x_1, \ldots, x_m$. We have assumed $r > 2(s-1)$ in the algorithm where $r = \text{up}(C)$ and $s = \text{side}(C)$. We now consider the possible forms of the stack K after the final side step. By the initialization step and Proposition 10, the bottom element of K is x_0. We consider the cases $s_p = 1$ and $s_p \geq 2$ separately.

Suppose $s_p = 1$, that is, the last sequence S_p of side steps consists of exactly one side step. Since x_0 is an upward starting point for C we then have $x_{m-2} \prec x_{m-1} \parallel x_m = x_0$. After the final ($p$th) up step, the element at the top of K is x_{m-1} and the element on the bottom is x_0. By Proposition 12, it follows that $x_0 \prec x_{m-1}$, a contradiction.

Now suppose $s_p \geq 2$, that is, the final sequence S_p of side steps contains at least two side steps. In the last (pth) side step we remove $\lambda_p = 2s_p - 1$ elements from the top of K without emptying it, $s_p - 1$ pairs of elements $b_1, \ldots, b_{2s_p-2}$ and then the single element $b_{2s_p-1} = e_{s_p-1}$ that is still at the top. So after the preceding (pth) up step the stack K consists of at least the $2s_p$ elements

$$b_{2s_p} \prec e_{s_p-1} \prec \cdots \prec b_1.$$

In addition, x_0 is on the bottom of the stack (and may equal b_{2s_p}). By (6) applied in the case $j = p$ and $i = s_p - 1$, it must be the case that

$$R(e_{s_p-1}) < R(d_{s_p-1}) \quad \text{and} \quad C(e_{s_p-1}) < C(d_{s_p-1}).$$

Because the representation is unit, the first inequality implies $L(e_{s_p-1}) < L(d_{s_p-1})$.

Since $b_{2s_p} \prec e_{s_p-1}$, we have $C(b_{2s_p}) < L(e_{s_p-1}) < L(d_{s_p-1})$. Similarly, $R(b_{2s_p}) < C(e_{s_p-1}) < C(d_{s_p-1})$. Thus, $b_{2s_p} \prec d_{s_p-1} \parallel d_{s_p} = x_0$. This contradicts the fact that either $x_0 = b_{2s_p}$ or $x_0 \prec b_{2s_p}$.

Since all possible forms of K after the last (pth) side step lead to a contradiction, it follows that $r \leq 2(s-1)$. This completes the proof of Theorem 6. □

3 Constructing split semiorders

In the preceding section, Theorem 6 gave an upper bound for the range of the wd_F function for split semiorders. Our goal in this section is to prove that for each rational number r/s for which r is in a range determined by s, there exists a split semiorder whose fractional weak discrepancy equals r/s. After stating this result, we outline the construction and give some preliminary lemmas before proving the theorem.

Theorem 13 *Let r, s be integers for which $s \geq 2$ and $s-1 \leq r \leq 2(s-1)$. There exists a split semiorder P with $wd_F(P) = r/s$ and an optimal forcing cycle C having* $\text{up}(C) = r,\ \text{side}(C) = s$.

Proof. When $r = s-1$ there exists a semiorder P that satisfies these conditions (Proposition 16 of [9]), and every semiorder is a split semiorder. Thus we may assume that $s \geq 2$ and $s \leq r \leq 2(s-1)$. We begin by constructing a unit point-core representation for a split semiorder $P = (V, \prec)$ possessing a forcing cycle C with $\text{up}(C) = r,\ \text{side}(C) = s$.

Begin by setting $V = \{x_0, x_1, \ldots, x_r, y_1, y_2, \ldots, y_{s-1}\}$. Define

$$q = \frac{r}{2s - r - 1}. \tag{8}$$

Notice that $2s - r - 1 = 2(s-1) - r + 1 \geq 1$, since we have assumed $2(s-1) \geq r$. Also $2s - r - 1 \leq 2r - r - 1 = r - 1$ since $s \leq r$. Thus,

$$1 < \frac{r}{r-1} \leq q \leq r.$$

For $0 \leq i \leq r$, define $I(x_i) = [L(x_i), R(x_i)]$ with splitting point $C(x_i)$ by

$$L(x_i) = i(q+1), \quad C(x_i) = i(q+1) + q, \quad R(x_i) = i(q+1) + 2q.$$

Similarly, for $1 \leq j \leq s-1$ let

$$L(y_j) = 2jq, \quad C(y_j) = 2jq, \quad R(y_j) = 2(j+1)q.$$

This collection of intervals $I(x_i), I(y_j)$ and splitting points $C(x_i), C(y_j)$ gives a representation of a split semiorder. Note that all the intervals have length $2q > 2$, and that the splitting point of $I(x_i)$ is at its midpoint while that of $I(y_j)$ is at its left endpoint.

Figure 3 illustrates the representation when $r = 6$ and $s = 4$, so $q = 6$. The labeling function g shown in the figure is described below.

It is straightforward to check that consecutive x-intervals are related by

$$\begin{aligned} L(x_{i+1}) &= C(x_i) + 1 = R(x_i) - (q-1) \\ C(x_{i+1}) &= R(x_i) + 1 \\ R(x_{i+1}) &= R(x_i) + q + 1 \\ L(x_{i+2}) &= R(x_i) + 2 \end{aligned} \tag{9}$$

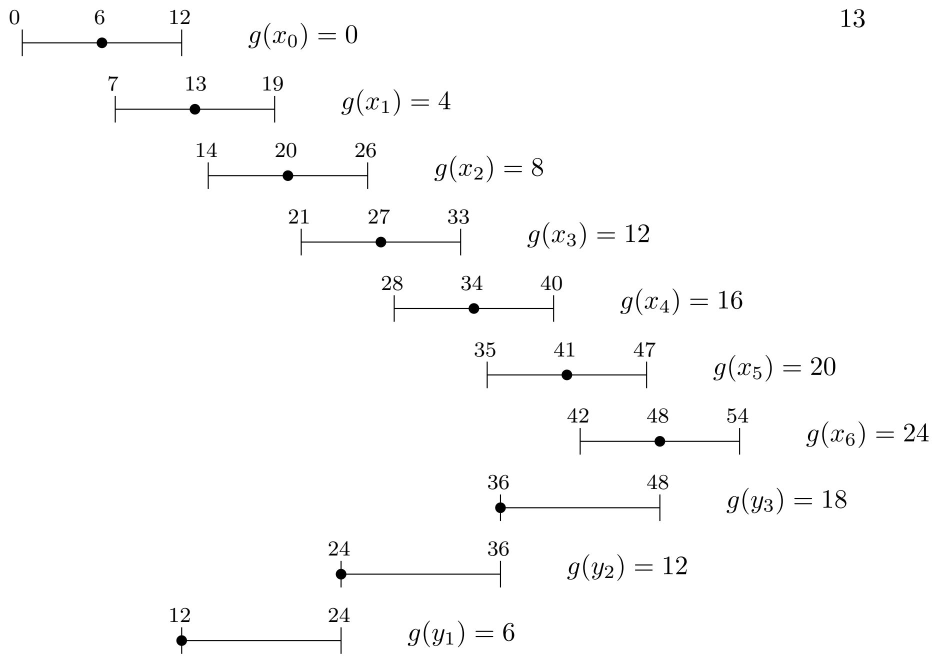

Figure 3: A unit point-core representation for a split semiorder together with a labeling function g.

It will be useful in what follows to express these endpoints and splitting points in terms of r and s. By (8), we have

$$\begin{aligned} L(x_i) &= i(q+1) = i\left(\frac{r}{2s-r-1}+1\right) = \frac{2is-i}{2s-r-1} \\ C(x_i) &= i(q+1)+q = \frac{2is-i+r}{2s-r-1} \\ R(x_i) &= i(q+1)+2q = \frac{2is-i+2r}{2s-r-1} \\ L(y_j) &= C(y_j) = 2jq = \frac{2jr}{2s-r-1} \\ R(y_j) &= 2(j+1)q = \frac{2jr+2r}{2s-r-1}. \end{aligned} \tag{10}$$

By construction, $x_i \prec x_{i+1}$ for $0 \leq i < r$ and $y_j \parallel y_{j+1}$ for $1 \leq j \leq s-2$. Furthermore, $y_1 \parallel x_0$ since $R(x_0) = 2q = C(y_1)$, and $x_r \parallel y_{s-1}$ since (10) implies

$$C(x_r) = \frac{2rs}{2s-r-1} = 2sq = R(y_{s-1}).$$

Thus, P contains the forcing cycle $C : x_0 \prec x_1 \prec \cdots \prec x_r \parallel y_{s-1} \parallel y_{s-2} \parallel \cdots \parallel y_1 \parallel x_0$, with $\text{up}(C) = r$ and $\text{side}(C) = s$. Theorem 5 implies that $wd_F(P) \geq r/s$.

We require a labeling of the elements of P that satisfies Definition 3 with $k = r/s$. Let

$$g(x_i) = is, \quad i = 0, 1, \ldots, r$$
$$g(y_j) = jr, \quad j = 1, 2, \ldots, s-1.$$

Then define the labeling $f : V \to \mathbf{Q}$ by $f(u) = g(u)/s$, i.e.,

$$f(x_i) = i$$
$$f(y_j) = j\frac{r}{s}.$$

For example, in the split semiorder P shown in Figure 3, $C : x_0 \prec x_1 \prec \cdots \prec x_6 \parallel y_3 \parallel y_2 \parallel y_1 \parallel x_0$ is a forcing cycle. Since $\text{up}(C) = 6, \text{side}(C) = 4$, Theorem 5 implies $wd_F(P) \geq 6/4$. The values of $g(u)$ are shown in the figure and $f(u) = g(u)/4$ satisfies Definition 3 with $k = 6/4$. Thus $wd_F(P) \leq 6/4$ and by combining the two inequalities we see $wd_F(P) = 6/4 = 3/2$.

To prove f satisfies Definition 3 it suffices to prove

(i) if $a \prec b$ then $g(a) + s \leq g(b)$ ("up" constraints)

(ii) if $a \parallel b$ then $|g(a) - g(b)| \leq r$. ("side" constraints)

We start with the "side" constraints (ii), as they are easier to prove. Since $x_i \parallel x_j$ if and only if $i = j$, there are only two cases to consider.

Case ($\parallel yy$). It is straightforward to check that when $i \geq j$, then $y_j \parallel y_i$ if and only if $i = j+1$ and that $|g(y_{j+1}) - g(y_j)| = r$.

Case ($\parallel xy$). We will prove that $|g(x_i) - g(y_j)| \leq r$ whenever x_i is incomparable to y_j. Suppose $x_i \parallel y_j$. By Lemma 11(a),

$$2jq = L(y_j) \leq R(x_i) = i(q+1) + 2q. \tag{11}$$

Also, either $L(x_i) \leq C(y_j) = L(y_j)$ or $C(x_i) \leq R(y_j)$. Thus, either

$$i(q+1) \leq 2jq \quad \text{or} \quad i(q+1) + q \leq 2(j+1)q = 2jq + 2q$$

and so in any case

$$i(q+1) \leq 2jq + q.$$

Combining this with (11) we obtain

$$i(q+1) \leq (2j+1)q \leq i(q+1) + 3q = (i+3)q + i.$$

Substituting $q = \frac{r}{2s-r-1}$ from (8) and noting that $i \leq r$, we find

$$i\left(\frac{r}{2s-r-1} + 1\right) \leq (2j+1)\frac{r}{2s-r-1} \leq (i+3)\frac{r}{2s-r-1} + i$$
$$ir + 2is - ir - i \leq (2j+1)r \leq (i+3)r + 2is - ir - i$$
$$2is - i \leq 2jr + r \leq 3r + 2is - i$$
$$-r - i \leq 2jr - 2is \leq 2r - i$$
$$-r \leq -\frac{r+i}{2} \leq jr - is \leq r - \frac{i}{2} \leq r.$$

This proves $|g(x_i) - g(y_j)| = |is - jr| \le r$, as desired.

We now return to the "up" constraints (i), where we want to show that if $a \prec b$ then $g(a) + s \le g(b)$.

Case ($\prec xx$). Let $x_i \prec x_k$, i.e., $i < k$. Then $g(x_i)+s = (i+1)s \le ks = g(x_k)$.

Case ($\prec yy$). Let $y_j \prec y_l$, i.e., $j \le l-2$. Since $s \le r$,

$$g(y_j) + s = jr + s \le (l-2)r + r = (l-1)r < lr = g(y_l).$$

In the remaining two "up" cases $\prec xy, \prec yx$ constraint (i) may not always be true, and it may therefore be necessary to alter slightly some of the intervals in the representation. This will change the poset $P = (V, \prec)$ by removing some comparabilities between pairs of elements and may also destroy the unit property of the representation. However, we will show that the new representation is proper, so Remark 2 will imply that the resulting poset $P' = (V, \prec')$ is a split semiorder. We will remove comparabilities in a way that will not affect any other pair of elements, so the conclusions we drew in the four cases considered so far will remain valid. We will see that P' satisfies properties (i) and (ii) of Definition 3 for all pairs of elements, so it will have the properties required by Theorem 13. We now consider the two "up" cases that remain.

Case ($\prec xy$). We must now consider all relations of the form $x_i \prec y_j$. We proceed by sweeping through the intervals $I(x_i)$ from right to left, i.e., with $i = r, r-1, \ldots, 1, 0$. For a given i, suppose $x_i \prec y_j$ for some y_j. Either we will prove that (i) is true or else we will redefine $L(y_j)$ and $C(y_j)$ by moving them to the left in a way that satisfies the constraints. This change will not affect the validity of the constraints for any i previously considered, i.e., for any larger value of i, so we may continue moving from right to left even when we modify the representation.

We first show that $i \le r-2$, i.e., this case cannot occur in the first two steps at the start of the sweeping process. Since the right endpoints of the x-intervals and the splitting points of the y-intervals are strictly increasing, it suffices to show that $R(x_{r-1}) \ge C(y_{s-1})$ and thus $x_{r-1} \not\prec y_{s-1}$. By (10) we have

$$C(y_{s-1}) = r\left(\frac{2s-2}{2s-r-1}\right),$$
$$R(x_{r-1}) = \frac{(r-1)(2s-1)+2r}{2s-r-1} = \frac{r(2s-2)+(3r-2s+1)}{2s-r-1}.$$

Since we have assumed $2 \le s \le r$ we know $3r - 2s + 1 \ge s + 1 \ge 3$. So $i \le r-2$.

Next we establish that

$$g(x_i) = is < jr = g(y_j). \tag{12}$$

Since $x_i \prec y_j$ we have $R(x_i) < C(y_j) = L(y_j)$, i.e. by (10)

$$R(x_i) = \frac{2is - i + 2r}{2s-r-1} < \frac{2jr}{2s-r-1} = C(y_j).$$

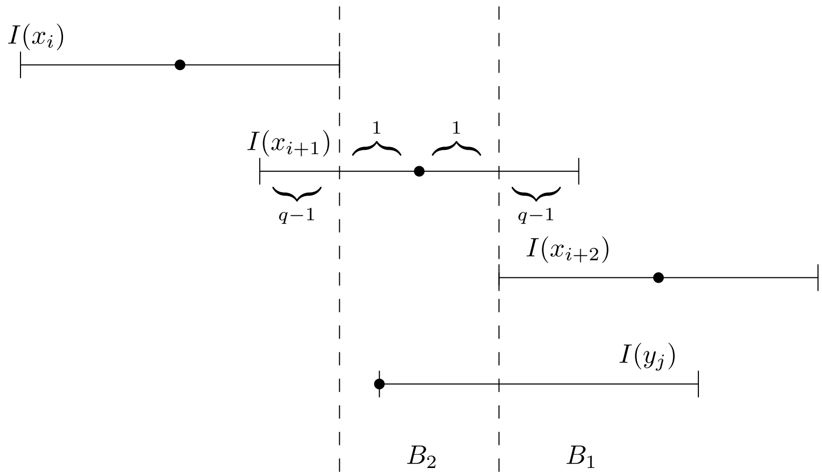

Figure 4: Case ($\prec xy$). If $L(y_j) \in B_2$ and $g(x_i) + s > g(y_j)$, slide $L(y_j)$ and $C(y_j)$ to the left to meet $R(x_i)$.

Since $2s - r - 1 > 0$ (see the sentence following (8)),

$$2is - i + 2r < 2jr.$$

Dividing by 2 and noting that $i < r$, we obtain $is < jr$.

Since $i \leq r - 2$, we know x_{i+2} is defined. There are now two subcases to consider depending upon whether or not the left endpoint of $I(y_j)$ lies to the right of the left endpoint of $I(x_{i+2})$. These are illustrated in Figure 4 by the regions B_1, B_2.

Region B_1. Suppose $L(y_j) \geq L(x_{i+2})$, i.e., the left endpoint of $I(y_j)$ is in the interval $B_1 = [L(x_{i+2}), L(y_{s-1})]$. By (10),

$$\frac{2jr}{2s - r - 1} \geq \frac{2(i+2)s - (i+2)}{2s - r - 1} = \frac{(i+2)(2s-1)}{2s - r - 1}.$$

Thus

$$(i+2)(2s-1) \leq 2jr,$$

i.e.,

$$(2is + 2s) + (2s - i - 2) \leq 2jr.$$

Now $i < r \leq 2(s-1)$ implies that $2s - i - 2 > 0$, so

$$g(x_i) + s = is + s < jr = g(y_j).$$

Region B_2. Now suppose $L(y_j) < L(x_{i+2})$, as illustrated in Figure 4. Since $R(x_i) < C(y_j) = L(y_j)$, the left endpoint of $I(y_j)$ is in the interval $B_2 = (R(x_i), L(x_{i+2}))$. If $g(x_i) + s \leq g(y_j)$, we are done with this subcase. Otherwise, slide the left endpoint and splitting point of $I(y_j)$ to the left until they meet the right endpoint of $I(x_i)$, i.e., replace $I(y_j)$ by the interval $I'(y_j)$ with $L'(y_j) =$

$C'(y_j) = i(q+1) + 2q$ and $R'(y_j) = R(y_j)$. We continue sweeping from right to left until we have considered each x_i in turn and modified the y-intervals in this way as needed.

All other intervals in the representation are unchanged, i.e., for all other $u \in V, I'(u) = [L'(u), R'(u)] = I(u)$. Also, the labeling of all elements of V is unchanged. This defines a new poset $P' = (V, \prec')$ where $\prec'$ is defined as in Definition 1.

We need to determine which relations in P can change in moving to P'. Since $C'(y_j) = R'(x_i)$ when $I(y_j)$ is modified, the corresponding relation $x_i \prec y_j$ becomes $x_i \parallel' y_j$. We will show these are the only relations that change.

By (9), the intervals B_2 are disjoint from one another for different x_i and the length of each one is 2. Also, the length of each y-interval before modification is $2q > 2$ and modifying it extends it only as far as the left endpoint of the corresponding B_2. So for each x_i at most one y_j can fall into this subcase, and each y_j falls into it for at most one x_i.

Remark 14 *Since the open interval B_2 does not contain the right endpoint of any x-interval, if $R(x_k) \leq L(y_j)$ then $R'(x_k) \leq L'(y_j)$. That is, if we move the left endpoint of a y-interval it does not pass the right endpoint of any x-interval.*

Let $I(y_j)$ be modified for some x_i. The only intervals whose endpoints or splitting points lie in $B_2 \cup \{R(x_i)\} = [R(x_i), L(x_{i+2}))$ are $I(x_i), I(x_{i+1}), I(y_j)$, and if $j \geq 2$ also $I(y_{j-1})$. So the only other relations that could change involve x_{i+1} or y_{j-1} together with y_j.

Before the move $y_j \parallel x_{i+1}$, since (9) implies

$$L(x_{i+1}) < R(x_i) < C(y_j) = L(y_j) < L(x_{i+2}) < R(x_{i+1}),$$

i.e., $C(y_j) \in I(x_{i+1})$. After the move $y_j \parallel' x_{i+1}$, since

$$L'(x_{i+1}) = L(x_{i+1}) < R(x_i) = C'(y_j) < R(x_{i+1}) = R'(x_{i+1}),$$

i.e., $C'(y_j) \in I'(x_{i+1})$.

Now let $j \geq 2$, so that y_{j-1} is defined. Before the move, $y_{j-1} \parallel y_j$. The splitting point $C(y_j) = R(y_{j-1})$ slides to the left at most 2 units but the length of $I(y_{j-1})$ is greater than 2, so after the move

$$L'(y_{j-1}) < C'(y_j) < R'(y_{j-1}).$$

Thus $C'(y_j) \in I'(y_{j-1})$ and so $y_{j-1} \parallel' y_j$.

So there is only one kind of new relation in P', namely, $x_i \parallel' y_j$. We must verify constraint (ii) for this new incomparability. We have $g(x_i) + s > g(y_j)$ by assumption and $g(x_i) < g(y_j)$ by (12). Thus,

$$-r < 0 < g(y_j) - g(x_i) < s \leq r.$$

Because the labeling has not changed, constraints (i) and (ii) remain valid for all the pairs we have considered in this and the previous cases. Also, the forcing cycle C in P remains a forcing cycle in P'.

If we redefined any intervals then we only changed the lengths of y-intervals, so in this case the interval representation of P' is no longer unit. However we now argue that it is a proper representation. It suffices to show that none of the new intervals $I'(y_j)$ properly contains any of the other representing intervals. Since we do not shift $L(y_j)$ beyond $L(y_{j-1})$ and $L'(y_{j-1}) \leq L(y_{j-1})$, $I'(y_j)$ cannot properly contain any y-interval in P'. Let $k \geq i+2$. Since $L(y_j) < L(x_{i+2})$, we have $R'(y_j) = R(y_j) < R(x_{i+2}) \leq R(x_k)$. Thus $I'(y_j)$ cannot properly contain $I'(x_k) = I(x_k)$. Similarly, for $k \leq i+1$, we have $L(x_k) \leq L(x_{i+1}) < R(x_i) = L'(y_j)$. So $I'(y_j)$ cannot properly contain $I(x_k)$ for any value of k. Thus, the resulting representation is proper and so P' is a split semiorder.

Case ($\prec yx$). For simplicity, we now let $P = (V, \prec)$ denote the poset obtained at the end of the preceding case, i.e., the split semiorder given by a proper representation.

We again sweep through the x-intervals from right to left. For a given x_i, suppose $y_j \prec x_i$ for some y_j. Either we will prove that (i) is true or else we will redefine $C(x_i)$ by moving it to the left. As before, for each x_i this will be the only relation that can change.

We first note that $i \geq 1$, i.e., this case cannot occur with the leftmost x-interval $I(x_0)$. This follows because for each y_j we have $C(y_j) \geq R(x_0) > L(x_0)$, i.e., $y_j \not\prec x_0$.

We next show
$$g(y_j) = jr < is = g(x_i). \tag{13}$$
Since $R(y_j) < C(x_i)$ and these points were not modified in the preceding case, it follows from (10) that
$$\frac{2jr + 2r}{2s - r - 1} < \frac{2is - i + r}{2s - r - 1}.$$
Thus
$$jr + r < is - \frac{i - r}{2},$$
and so
$$jr < jr + \frac{r + i}{2} < is.$$

There are once again two subcases to consider depending upon whether or not the right endpoint of $I(y_j)$ lies to the left of the right endpoint of $I(x_{i-1})$. Note that since $i \geq 1$, we know x_{i-1} is defined. As before, we will either prove that the "up" constraint (i) is true or redefine the poset P accordingly. While we omit the picture, this situation can be illustrated in a way analogous to Figure 4.

Region D_1. Suppose $R(y_j) \leq R(x_{i-1})$, i.e., the right endpoint of $I(y_j)$ is in the interval $D_1 = [R(y_1), R(x_{i-1})]$. By (10),
$$\frac{2jr + 2r}{2s - r - 1} \leq \frac{(i-1)(2s-1) + 2r}{2s - r - 1}$$

and thus

$$2jr \leq (i-1)(2s-1) = 2is - i - 2s + 1.$$

Since $i \geq 1$ we have $2jr + 2s \leq 2is - i + 1 < 2is + 1$, and since both r and s are integers this implies $2jr + 2s \leq 2is$. Therefore

$$g(y_j) + s \leq g(x_i),$$

and (i) is true for this subcase.

Region D_2. Now suppose $R(y_j) > R(x_{i-1})$. Since $R(y_j) < C(x_i)$, the right endpoint of $I(y_j)$ is in the interval $D_2 = (R(x_{i-1}), C(x_i))$. If $g(y_j) + s \leq g(x_i)$, we are done with this subcase. Otherwise, redefine $C(x_i)$ by sliding it to the left to equal $R(y_j)$, i.e., let $C'(x_i) = R(y_j)$. We continue sweeping from right to left, taking each x_i in turn and moving the splitting points $C(x_i)$ as needed. All endpoints and labels remain unchanged. This defines a new poset $P' = (V, \prec')$. We will prove P' has the properties sought in Theorem 13.

For the relation $\prec'$ to define P' as a split semiorder, we must verify that $C'(x_i) \in I'(x_i)$ for each x_i. First note that since $q > 1$, the assumptions of this subcase and (9) imply that $C(x_i) = L(x_i) + q$ is moved to the left by less than $|D_2| = C(x_i) - R(x_{i-1}) = 1 < q$ and so is still in $I(x_i)$. That is, after the shift we have $C'(x_i) \in I'(x_i)$.

Since the representation in the preceding case was proper and only the splitting points in the intervals changed, this representation is also proper and P' is a split semiorder. Since $C'(x_i) \in I'(y_j)$, we know $y_j \parallel' x_i$. We will show these are the only relations that change in moving to P'.

By (9), the intervals D_2 are disjoint from one another for different x_i. The right endpoints of consecutive y-intervals are $2q > 2$ units apart whether or not the left endpoints were changed in the preceding case. So for a given x_i at most one y_j can fall into this subcase. Also, modifying $C(x_i)$ extends it only as far as the left endpoint of the corresponding D_2, so each y_j falls into this subcase for at most one x_i.

Let some $C(x_i)$ be modified. The only intervals whose endpoints or splitting points lie in $D_2 \cup \{R(x_{i-1})\} = [R(x_{i-1}), C(x_i))$ are $I(x_{i-1}), I(x_i), I(y_j)$, and if $j \leq s-2$ also $I(y_{j+1})$. So the only other relations that could change involve x_{i-1} or y_{j+1} together with x_i.

Since $i \geq 1$, x_{i-1} is defined and $x_{i-1} \prec x_i$ before the move. The splitting point $C(x_i)$ slides to the left but not as far as the right endpoint of $I(x_{i-1})$, so after the move $x_{i-1} \prec' x_i$.

Now let $j \leq s-2$, so that y_{j+1} is defined. Whether or not we modified $L(y_{j+1})$ in Case ($\prec xy$), Remark 14 and (9) imply that in the current case

$$R(x_{i-1}) \leq L(y_{j+1}) \leq R(y_j) < C(x_i) = R(x_{i-1}) + 1 \leq L(y_{j+1}) + 1 < R(y_{j+1}).$$

Thus $C(x_i) \in I(y_{j+1})$ and so $x_i \parallel y_{j+1}$ before the move. Modifying $C(x_i)$ only slides it as far as $R(y_j)$, so after the move $C'(x_i) \in I'(y_{j+1})$ and $x_i \parallel' y_{j+1}$.

Therefore, the only changes in the partial ordering can be from $y_j \prec x_i$ to $y_j \parallel' x_i$.

Next, we must prove (ii) holds for $y_j \parallel' x_i$. Since $g(y_j) + s > g(x_i)$ by the assumptions of this subcase and $g(y_j) < g(x_i)$ by (13), we have

$$-r < 0 < g(x_i) - g(y_j) < s \leq r.$$

So the constraints (i) and (ii) hold for all the pairs we have considered in this and the previous cases. The forcing cycle C in P remains a forcing cycle in P'.

Finally, since $\text{up}(C) = r, \text{side}(C) = s$, Theorem 5 implies $wd_F(P') \geq r/s$. On the other hand, the labeling f constructed in the proof shows $wd_F(P') \leq r/s$, so we conclude $wd_F(P') = r/s$. This completes the proof of Theorem 13. □

The following examples show that we may indeed need to modify the partial ordering as we did in the final two cases in Theorem 13. One can check that when $r = 7, s = 6$ then in Case ($\prec xy$) we have $x_5 \prec y_5$ and must slide $L(y_5)$ and $C(y_5)$ to meet $R(x_5)$. In Case ($\prec yx$) we have $y_1 \prec x_2$ and must slide $C(x_2)$ to meet $R(y_1)$.

Finally, we can combine Theorems 6 and 13 to describe the range of the fractional weak discrepancy function for split semiorders.

Corollary 15 *For any rational number $q > 0$, there exists a split semiorder P with $wd_F(P) = q$ if and only if $q = r/s$ for some integers r, s with $0 \leq s - 1 \leq r < 2s$.*

Proof. Suppose P is a split semiorder with $wd_F(P) = q$. If $q = 0$ we let $r = 0,\ s = 1$. If $q > 0$ then P has an incomparable pair and thus has an optimal forcing cycle C. Letting $r = \text{up}(C)$ and $s = \text{side}(C)$, we have $s \geq 2$ and $q = r/s$. By Theorem 6, $r \leq 2(s-1)$. By [9], Proposition 17, either $r = s - 1$ (in which case P is a semiorder) or $r \geq s$. Thus $1 \leq s - 1 \leq r \leq 2(s-1) < 2s$ and so in all cases q has the desired representation.

Conversely, suppose $q = r/s$, where $0 \leq s - 1 \leq r < 2s$. If $s = 1$ and $q = r = 0$, we can let P be any linear order. If $s = 1$ and $q = r = 1$, we can let $P = \mathbf{3} + \mathbf{2}$. For $s \geq 2$, suppose first that $r \leq 2(s-1)$. Then by Theorem 13 there is a split semiorder P with $wd_F(P) = q$ and having an optimal forcing cycle C with $\text{up}(C) = r,\ \text{side}(C) = s$. Otherwise, $r = 2s - 1$ and by Theorem 6 there is no split semiorder with such a forcing cycle. In this case we let $r' = 2r,\ s' = 2s$. We will show that $r',\ s'$ satisfy the hypotheses of Theorem 13. Since $s \geq 2$ we have $r \geq s$, and so $0 \leq 2s - 1 \leq 2r - 1 < 2r = 2(2s-1)$. Thus $0 \leq s' - 1 \leq r' = 2(s'-1)$, and so there is a split semiorder P with $wd_F(P) = r'/s' = q$ and having an optimal forcing cycle C with $\text{up}(C) = r',\ \text{side}(C) = s'$. □

Corollary 15 can be used to extend the scope of Theorems 6 and 13. For example, by Theorem 6 there is no split semiorder P with $wd_F(P) = 3/2$ that has an optimal forcing cycle C with $r = \text{up}(C) = 3,\ s = \text{side}(C) = 2$. But by Corollary 15 there is a split semiorder P with $wd_F(P) = 3/2$ having an optimal forcing cycle C with $r' = \text{up}(C) = 6$ and $s' = \text{side}(C) = 4$. In fact, Figure 3 gave an example of such a split semiorder.

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