Total Linear Discrepancy for the sum of two chains

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ABSTRACT

In this paper we introduce the notion of the *total linear discrepancy* of a poset as a way of measuring the fairness of linear extensions. For posets that are the sum of two chains, we characterize those linear extensions that are optimal with respect to total linear discrepancy. The characterization provides an easy way to count the number of optimal linear extensions.

1 Introduction

We begin with some definitions and notation. In this paper we consider only finite posets. A poset $P = (X, \prec)$ consists of a ground set X together with an order relation $\prec$. If there are several posets under consideration, we write $\prec_P$. When points $x, y, \in X$ are incomparable we write $x \parallel_P y$ or just $x \parallel y$. If there are no incomparabilities then P is a *linear order* or *chain*. If $x \prec y$ and there is no $z \in X$ with $x \prec z \prec y$ we say y *covers* x and write $x \prec: y$. A *linear extension* L of a poset P is a linear order that respects the relation of P, that is, $x \prec_L y$ whenever $x \prec_P y$. The *height* of a point x in a linear order L, denoted by $h_L(x)$, is the greatest cardinality of a chain whose maximum point is x. Finally, the poset $\mathbf{r} + \mathbf{s}$ consists of the disjoint union of a chain of r points with a chain of s points. For all terminology and notation not defined here, we refer the reader to [9].

The linear discrepancy of a poset was introduced by Tanenbaum, Trenk and Fishburn [8] as a measure of how far a poset is from being a linear order. It was studied further in [1], [3], [4] and [5]. Formally,

$$ld(P) = \min_{L} \max_{x \| y} |h_L(x) - h_L(y)|$$

where the minimum is taken over all linear extensions L of P.

The concept of linear discrepancy arises in many real world problems where a linear extension of a poset is required and it is desirable to choose one that minimizes the difference in height of incomparable points. For example, a poset could represent a set of projects on a professor's desk, ordered by urgency (or perhaps importance). The professor must tackle these projects one at a time and should order them so that more urgent projects are done before less urgent ones (i.e., a linear extension). In addition, if two projects are incomparable, the professor would not want to complete them at widely different times, for this could be viewed as unfair by the person awaiting the completion of the second project. Thus the professor seeks a linear extension that achieves the linear discrepancy for this poset. Other examples appear in [8].

In this paper we consider a different measure of fairness. Rather than seeking to minimize the *maximum* difference in height between incomparable elements, we now seek to minimize the *average* such difference. Equivalently, we seek to minimize the *sum* of such differences.

Definition 1 The *total linear discrepancy* of a poset P is

$$tld(P) = \min_{L} \sum_{x \|_P y} |h_L(x) - h_L(y)|.$$

where the minimum is taken over all linear extensions L of P. A linear extension for which this minimum value is achieved is called *optimal*.

It will sometimes be useful to refer to the sum in Definition 1 for a particular linear extension.

Definition 2 Let P be a poset and L be a linear extension of P. The *total discrepancy of P in L*, denoted by $td_P(L)$, as is

$$td_P(L) = \sum_{x \|_P y} |h_L(x) - h_L(y)|.$$

Thus $tld(P) = \min_L td_P(L)$ where the minimum is taken over all linear extensions L of P.

In this paper we focus on the posets of the form $\mathbf{r} + \mathbf{s}$ and determine which linear extensions are optimal with respect to total linear discrepancy. A linear extension of the poset $\mathbf{r} + \mathbf{s}$ can be viewed as a linear arrangement of r X's (the points of the r-chain in their natural order) and s O's (the points of the s-chain in their natural order). All the figures in this paper denote linear extensions of $\mathbf{r}+\mathbf{s}$ in this way.

Posets of the form $\mathbf{r}+\mathbf{s}$ play a central role in linear discrepancy and the related concept of weak discrepancy. They form the basis for forcing cycles which are the key tool in studying weak discrepancy ([2]), provide lower bounds on both linear and weak discrepancy for posets that possess them, and in several cases provide upper bounds on weak discrepancy for posets that do not possess certain $\mathbf{r}+\mathbf{s}$ combinations (e.g, see [6], [7]).

For linear discrepancy, optimal extensions of $\mathbf{r}+\mathbf{s}$ have the smaller of the two chains inserted in the middle of the larger [8]. This results in four optimal linear extensions (when r and s are equal and odd), two optimal linear extensions (when r and s are equal and even or when $r < s$ and s odd), or a unique linear extension (when $r < s$ and s even). In contrast, we will see in Corollary 14 that there are 2^r linear extensions that are optimal with respect to total linear discrepancy when $r \leq s$ and $r+s$ is even and a unique optimal linear extension when $r+s$ is odd.

The next two definitions help in calculating the effect of swapping two consecutive points in a linear extension of $\mathbf{r}+\mathbf{s}$.

Definition 3 Let L be a linear extension of the poset $P = \mathbf{r}+\mathbf{s}$. For any point γ in P, let $down(\gamma)$ be the number of points in P below γ in L that are comparable to γ in P and similarly, let $up(\gamma)$ be the number of points in P above γ in L that are comparable to γ in P

Definition 4 Let L be a linear extension of poset $P = \mathbf{r}+\mathbf{s}$ and let α and β be incomparable elements of P that appear consecutively in L as $\alpha \prec_L \beta$. Then we define the difference (Δ) of this pair to be $\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta)$.

Our primary tool for demonstrating that a linear extension L is not optimal will be to produce two elements α and β that are consecutive in L and show that swapping them results in a lower total linear discrepancy. The next lemma facilitates this.

Lemma 5 *Let L_1 and L_2 be linear extensions of a poset $P = \mathbf{r}+\mathbf{s}$ that are identical except for one pair of points $\alpha \parallel \beta$ in P that appears as $\alpha \prec\!: \beta$ in L_1 and $\beta \prec\!: \alpha$ in L_2. If Δ of this pair is defined as in Definition 4, then $td_P(L_2) = td_P(L_1) + \Delta$.*

Proof. Without loss of generality, assume α is from the $\mathbf{r}$-chain and β is from the $\mathbf{s}$-chain. Swapping α and β brings α one unit farther from the $down(\beta)$ elements of the $\mathbf{s}$-chain below it in L_1 and likewise brings β one unit farther from the $up(\alpha)$ elements of the $\mathbf{r}$-chain above it in L_1. At the same time, it brings α one unit closer to the $up(\beta)$ elements of the $\mathbf{s}$-chain above it in L_1, and brings β one unit closer to the $down(\alpha)$ elements of the $\mathbf{r}$-chain below it in L_1. Thus in moving from L_1 to L_2, we add $down(\beta)+up(\alpha)-up(\beta)-down(\alpha) = \Delta$ to the total discrepancy of P in L_1. □

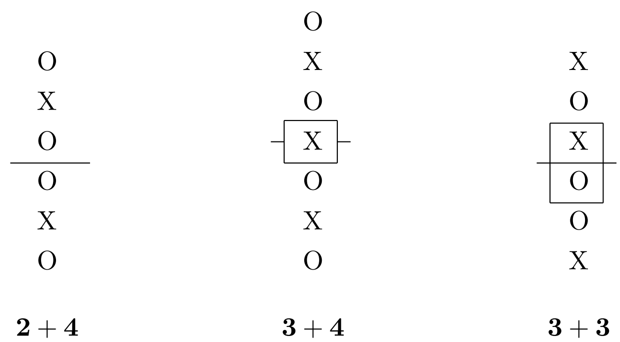

Figure 1: The hub of each linear extension is shown in a (perhaps empty) box. The elements of the **r**-chain are depicted by X's and those of the **s**-chain by O's.

The next definition identifies a center portion of a linear extension, called the hub, in a way that allows for an equal distribution of points from each chain above and below the hub.

Definition 6 Let $P = \mathbf{r} + \mathbf{s}$ and L be a linear extension of P. The *hub* of L consists of 0, 1, or 2 points as follows:

- If r and s are both even, the hub of L consists of no points.
- If $r + s$ is odd, the hub of L consists of 1 point.
- If r and s are both odd, the hub of L consists of 2 points.

In all cases, the hub is located exactly in the center of L.

We will see below in Proposition 8 that in the case of an optimal linear extension, the hub will consist of a point from each chain that is odd. Figure 1 shows examples of each of these cases.

2 Optimal Extensions for Posets $\mathbf{r} + \mathbf{s}$

In this section we characterize those linear extensions of $\mathbf{r} + \mathbf{s}$ that are optimal with respect to total linear discrepancy.

Proposition 7 *In any optimal linear extension of $P = \mathbf{r} + \mathbf{s}$ there will never be three consecutive points from the same chain, except possibly at the ends.*

Proof. Suppose for a contradiction we had an optimal linear extension L of P consisting of a set R of $n \geq 3$ points from the r-chain appearing consecutively, flanked by points from the s-chain. Let α_1 be the highest point from R in L and β_1 be the point just above it in L (which must be from the s-chain). Similarly, let β_2 be the lowest point from R in L and α_2 be the point just below it in L (which must also be from the s-chain). Let x_1 (resp. x_2) be the number of

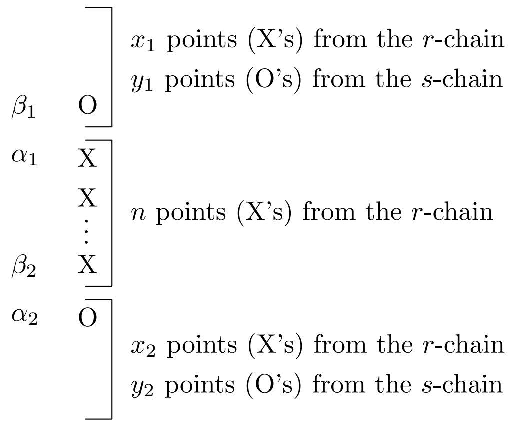

Figure 2: A linear extension in which there are $n \geq 3$ points from one chain in a row, flanked by points from the other chain.

points from the r-chain above (resp. below) R in L and let y_1 (resp. y_2) be then number of points from the s-chain above (resp. below) R in L. Figure 2 illustrates the situation.

Then if α_1 and β_1 are swapped to get a new linear extension L' of P, the resulting difference is

$\Delta_1 = down(\beta_1) - down(\alpha_1) + up(\alpha_1) - up(\beta_1) = y_2 - (x_2 + n - 1) + x_1 - (y_1 - 1) = y_2 + x_1 - x_2 - y_1 + 2 - n.$

Similarly, if α_2 and β_2 are swapped to get a new linear extension L'' of P, the resulting difference is

$\Delta_2 = down(\beta_2) - down(\alpha_2) + up(\alpha_2) - up(\beta_2) = x_2 - (y_2 - 1) + y_1 - (x_1 + n - 1) = x_2 + y_1 - y_2 - x_1 + 2 - n.$

If we let $q = y_2 + x_1 - x_2 - y_1$ then we can rewrite $\Delta_1 = q + (2 - n)$ and $\Delta_2 = -q + (2 - n)$. Since we have assumed $n \geq 3$ we know $2 - n \leq -1$. Thus if $q \leq 0$ then $\Delta_1 \leq -1$ and $td_P(L') \leq td_P(L)$, a contradiction. Otherwise, if $q \geq 0$ then $\Delta_2 \leq -1$ and $td_P(L'') \leq td_P(L)$, a contradiction. $\square$

Next we specify which points can occupy the hub of an optimal linear extension of $\mathbf{r} + \mathbf{s}$.

Proposition 8 *Let $P = \mathbf{r} + \mathbf{s}$ and let L be an optimal linear extension of P. If $r + s$ is odd, the hub of L consists of 1 point from the odd chain. If r and s are both odd, the hub of L consists of 2 points, one from each chain.*

Proof. First consider the case in which $r + s$ is odd, and without loss of generality we may assume r is odd and s is even. Thus, $r \geq 1, s \geq 2$. By definition, the hub of L consists of one point, located at the center of L. We must show this point is from the r-chain, so for a contradiction, assume it is from the s-chain. Since r is odd, there must be at least $\frac{r+1}{2}$ points from the

r-chain either above or below the hub, so without loss of generality we may assume there are this many above the hub. Since there are a total of $\frac{r+s-1}{2}$ points in total above the hub, at most $\frac{s-2}{2}$ points from the s-chain lie above the hub. The remaining (at most $\frac{r-1}{2}$) points from the r-chain and (at least $\frac{s}{2}$) points from the s-chain lie below the hub.

Let β be the lowest point from the r-chain above the hub and α be the point from the s-chain directly below it in L. Note that α is a point either in or above the hub. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq \frac{r-1}{2} - \frac{s}{2} + \frac{s-2}{2} - (\frac{r+1}{2} - 1) = -1 < 0$. Thus L is not optimal, a contradiction.

Now consider the case in which r and s are both odd. The case $r = s = 1$ is trivial, so suppose that either $r \geq 3$ or $s \geq 3$. For a contradiction, assume both points in the hub are from the same chain, without loss of generality from the r-chain, so $r \geq 3$. Thus there are an odd number $(r-2)$ remaining points from the r-chain to be distributed above and below the hub. Without loss of generality we may assume there are at least $\frac{r-1}{2}$ from this chain above the hub and at most $\frac{r-3}{2}$ from the r-chain below the hub. There are a total of $\frac{r+s-2}{2}$ points above the hub, thus at most $\frac{s-1}{2}$ points above the hub can be from the s-chain and the remaining at least $\frac{s+1}{2}$ points from the s-chain must be below the hub.

Since $\frac{s+1}{2} \geq 1$ there is at least one point from the s-chain below the hub. Let α be the highest point from the s-chain below the hub and let β be the point from the r-chain directly above it. Note that β is either the lower of the two hub points or lies below the hub. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq \frac{r-3}{2} - (\frac{s+1}{2} - 1) + \frac{s-1}{2} - (\frac{r-1}{2} + 1) = -2 < 0$. Thus L is not optimal, a contradiction. $\Box$

By Proposition 8, in an optimal linear extension of $\mathbf{r} + \mathbf{s}$, there will always be an even number of points from each chain that are outside of the hub. A linear extension of $P = \mathbf{r} + \mathbf{s}$ is said to be *balanced around the hub* if there are an equal number of elements from the **r**-chain above and below the hub and there are an equal number of elements from the **s**-chain above and below the hub. Each of the linear extensions in Figure 1 is balanced around the hub.

Proposition 9 *Any optimal linear extension of $P = \mathbf{r} + \mathbf{s}$ is balanced around the hub.*

Proof. Let L be an optimal linear extension of P. We consider cases depending on the parity of r and s.

Case 1: $r + s$ is odd.

We may assume, without loss of generality, that r is odd and s is even. The case $r = 1$ follows immediately from Proposition 8, so we assume $r \geq 3$ and $s \geq 2$. By Proposition 8 we know that the hub of L consists of a single point from the r-chain. Thus there are an even number $(r-1)$ of points from the r-chain and an even number of points from the s-chain (s) remaining to be distributed above and below the hub.

If the set of points from one chain are not equally distributed around the hub, then neither are the points from the other chain. Thus for a contradiction, we may assume that the points from the r-chain are not distributed equally above and below the hub, and without loss of generality that at least $\frac{r+1}{2}$ points from the r-chain are above the hub. Then the remaining (at most $\frac{r-3}{2}$) points from the r-chain must lie below the hub. Since there are a total of $\frac{r+s-1}{2}$ points above the hub and the same number below, there are at most $\frac{s-2}{2}$ points from the s-chain above the hub and at least $\frac{s+2}{2}$ below.

Let α be the highest point from the s-chain below the hub and β be the point from the r-chain directly above α. Then β is either the point in the hub, or a point below it. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq \frac{r-3}{2} - \frac{s}{2} + \frac{s-2}{2} - \frac{r+1}{2} = -3 < 0$. Thus L is not optimal, a contradiction.

Case 2: r, s **are both odd.**

By Proposition 8 we know that the hub of L consists of two points, one from each chain. If either $r = 1$ or $s = 1$ the result follows from Proposition 8 and the definition of hub. Thus, we assume $r, s \geq 3$. Again there are an even number of points from each chain remaining to be distributed above and below the hub. As before, if the points from one chain are not distributed equally above and below the hub, then this necessitates that the points from the other chain are also distributed unequally. So for a contradiction we may assume, without loss of generality, that there are at least $\frac{r+1}{2}$ points from the r-chain are above the hub. Then the remaining (at most $\frac{r-3}{2}$) points from the r-chain must lie below the hub. Since there are a total of $\frac{r+s-2}{2}$ points above the hub and the same number below, there are at most $\frac{s-3}{2}$ points from the s-chain above the hub and at least $\frac{s+1}{2}$ below.

If the hub consists of a point from the s-chain below a point from the r-chain, then let α be the lower point in the hub and β be the higher point in the hub. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq \frac{r-3}{2} - \frac{s+1}{2} + \frac{s-3}{2} - \frac{r+1}{2} = -4 < 0$. Thus L is not optimal, a contradiction.

Otherwise, the hub consists of a point from the r-chain below a point from the s-chain. In this case, let β be the lowest point from the r-chain above the hub and α be the point from the s-chain directly below it. Note that by Proposition 7, the point α either lies in the hub or just above it. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq \frac{r-1}{2} - \frac{s+1}{2} + \frac{s-3}{2} - \frac{r-1}{2} = -2 < 0$. Thus again, L is not optimal, a contradiction.

Case 3: r, s **are both even.**

In this case, there are no points in the hub. Without loss of generality we may assume that more than half the points from the r-chain lie above the hub, thus at least $\frac{r}{2} + 1$ points from the r-chain are above the hub and the remaining (at most $\frac{r}{2} - 1$) points form that chain lie below the hub. As before, at most $\frac{s}{2} - 1$ points from the s-chain are above the hub and the remaining (at least $\frac{s}{2} + 1$) points form the s-chain lie below the hub.

Consider the lowest point from the r-chain above the hub. If the point directly below it is from the s-chain, then let these points respectively be β and α. The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq (\frac{r}{2} - 1) - \frac{s}{2} + (\frac{s}{2} - 1) - \frac{r}{2} = -2 < 0$.
Thus again, L is not optimal, a contradiction.

Otherwise, the points just above the hub and just below the hub must both be from the r-chain. Let α be the higheest point from the s-chain below the hub and β be the point directly above it, which must be from the r-chain. Now the difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq (\frac{r}{2} - 2) - \frac{s}{2} + (\frac{s}{2} - 1) - (\frac{r}{2} + 1) = -4 < 0$.
Thus again, L is not optimal, a contradiction. $\square$

In the case of $P = \mathbf{r} + \mathbf{s}$ where $r + s$ is even, there is a unique optimal linear extension with respect to total linear discrepancy, as given in the following theorem.

Theorem 10 *Let $P = \mathbf{r} + \mathbf{s}$ where $r + s$ is odd. Then there is a unique optimal linear extension L of P as follows: (i) the hub of L consists of one point from the odd chain, (ii) points from the r-chain and s-chain alternate from the hub outward (iii) any extra points from the larger chain are distributed equally at the ends of L.*

Proof. Without loss of generality, assume that r is odd and s is even. The case $r = 1$ follows from Proposition 8, so we assume $r \geq 3, s \geq 2$. Let L be an optimal linear extension of $P = \mathbf{r} + \mathbf{s}$. The claim in (i) follows directly from Proposition 8. Furthermore, Proposition 9 ensures that there will be $\frac{r-1}{2}$ points from the r-chain above and below the hub and $\frac{s}{2}$ points from the s-chain above and below the hub. We prove (ii) and (iii) by showing that it is impossible to have two or more points that are consecutive in L, come from the same chain, and are flanked by points from the other chain. By Proposition 7, we know there can not be three or more such consecutive points.

First assume there are two points from the r-chain that are consecutive in L, without loss of generality, occur at or above the hub, and are flanked by points from the s-chain. Consider the lowest occurrence of this at or above the hub. Thus we may assume there are $c \geq 0$ points from the s-chain alternating with c points from the r chain directly above the hub, followed by a point α from the r-chain. Let β be the point from the s-chain directly above α in L.

The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) = (\frac{s}{2} + c) - (\frac{r-1}{2} + c + 1) + (\frac{r-1}{2} - c - 1) - (\frac{s}{2} - c - 1) = -1 < 0$. Thus L is not optimal, a contradiction.

Now assume there are two points from the s-chain that are consecutive in L, without loss of generality, occur above the hub, and are flanked by points from the r-chain. Consider the lowest occurrence of this at or above the hub, thus we may assume there are $c \geq 0$ points from the s-chain alternating with c points from the r-chain starting just above the hub and going upwards in L, followed by two points from the s-chain. Let α be the higher of these two points and let β be the point from the r-chain directly above α in L.

The difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq (\frac{r-1}{2} + c + 1) - (\frac{s}{2} + c + 1) + (\frac{s}{2} - c - 2) - (\frac{r-1}{2} - c - 1) = -1 < 0$. Thus L is not optimal, a contradiction. □

When $r + s$ is even, there is more than one optimal linear extension for $P = \mathbf{r} + \mathbf{s}$, but as we show in Proposition 11, they all satisfy the following condition.

A linear extension of $P = \mathbf{r} + \mathbf{s}$ satisfies the *domino condition* if it has the following form.

- Reserve one point from each chain that is odd to form the hub.
- From the remaining points, form as many dominoes as possible, each consisting of one point from the **r**-chain and one from the **s**-chain. The points within a domino may appear in either order.
- Set the dominoes out above and below the hub, evenly distributed.
- Put the extra points on the ends, evenly distributed.

Proposition 11 *Any optimal linear extension of $P = \mathbf{r} + \mathbf{s}$ satisfies the domino condition.*

Proof. Let L be any optimal linear extension of $P = \mathbf{r} + \mathbf{s}$. By Proposition 8, the hub of L consists of one point from each odd chain. This leaves an even number of points from the r-chain and an even number of points from the s-chain to be distributed above and below the hub. The case in which $r + s$ is odd follows directly from Theorem 10.

Next consider the case in which $r + s$ is even. Suppose the domino condition is violated, that is, counting by pairs from the hub outwards, there is a pair of points from the same chain. Without loss of generality, we may assume the violation closest to the hub consists of two points from the r-chain and occurs above the hub. Let α be the higher of these two points and let $c \geq 1$ be the number of dominoes below this violation down to and including the hub.

By Proposition 7, the points from the r chain in this violation are flanked by points from the s-chain. Let β be the point from the s-chain directly above α. By Proposition 9, half the points from each chain are below the hub. Thus in the case of r and s both odd, the difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq$
$(\frac{s-1}{2} + 1 + c) - (\frac{r-1}{2} + c + 2) + (\frac{r-1}{2} - c - 2) - (\frac{s-1}{2} - c - 1) = -2 < 0$.

And in the case of r and s both even, the difference in swapping α and β is
$\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) \leq$
$(\frac{s}{2} + c) - (\frac{r}{2} + c + 1) + (\frac{r}{2} - c - 2) - (\frac{s}{2} - c - 1) = -2 < 0$.

Thus in either case, L is not optimal, a contradiction. □

Theorem 12 *Let $P = \mathbf{r} + \mathbf{s}$ be the sum of two chains. There exists a linear extension of P that is optimal with respect to total linear discrepancy and is balanced around the hub and satisfies the domino condition.*

Proof. We know that optimal linear extensions exist because there are only finitely many linear extensions of poset P. The proofs that an optimal linear extension is balanced around the hub (Proposition 9) and satisfies the domino condition (Proposition 11) proceed by taking a linear extension that violates one of the conditions and showing that two points can be swapped to yield a difference $\Delta < 0$. Thus linear extensions that are not balanced around the hub or that do not satisfy the domino condition can not be optimal and hence any optimal linear extension is balanced around the hub and satisfies the domino condition. $\square$

Theorem 13 *Let $P = \mathbf{r} + \mathbf{s}$ be the sum of two chains. If $r+s$ is even, any linear extension that is balanced around the hub and satisfies the domino condition is optimal with respect to total linear discrepancy. If $r+s$ is odd, there is a unique linear extension which is optimal with respect to total linear discrepancy; it is balanced around the hub, satisfies the domino condition, and the dominoes must be placed so that elements from different chains alternate, except possibly at the ends.*

Proof. The case in which $r+s$ is odd is already covered in Theorem 10, thus it remains to consider the case in which $P = \mathbf{r} + \mathbf{s}$ and $r+s$ is even. By Proposition 12, there exists a linear extension L of P that is optimal with respect to total linear discrepancy and is balanced around the hub and satisfies the domino condition. Let L' be any other linear extension of P that is balanced around the hub and satisfies the domino condition. If r and s are both even, consider the hub as an additional domino. Then by the definition of the domino condition, we can get from L' to L by a sequence of swaps of pairs of points within a domino. Thus it suffices to show that such a swap gives a difference (Δ) of 0.

Without loss of generality, we may assume $r \leq s$. Let α be the lower point in a domino and β the point just above it in the same domino. Then below this domino are $x \geq 0$ dominoes, consisting of x points from the r-chain and x points from the s-chain and below that, $(s-r)/2$ points from the s-chain. Similarly, above the domino are $y = r - x - 1$ points from the each chain and above them, the remaining $(s-r)/2$ points from the s-chain. The difference in swapping α and β is $\Delta = down(\beta) - down(\alpha) + up(\alpha) - up(\beta) = x - x + y - y + (s-r)/2 - (s-r)/2 = 0$. $\square$

Since a linear extension of a poset $\mathbf{r} + \mathbf{s}$ corresponds to a linear arrangement of r X's and s O's, the number of such linear extensions is $\binom{r+s}{r} = \frac{(r+s)!}{r!s!}$. Using Theorem 13, we can calculate the number of these that are optimal with respect to total linear discrepancy.

Corollary 14 *Let $P = \mathbf{r} + \mathbf{s}$ where $r \leq s$. If $r+s$ is odd, then P has a unique optimal linear extension with respect to total linear discrepancy, and if $r+s$ is even, then P has 2^r optimal linear extensions with respect to total linear discrepancy.*

Proof. The case where $r+s$ is odd follows immediately from Theorem 13. In the case where $r+s$ is even, there will be r dominoes and two possible

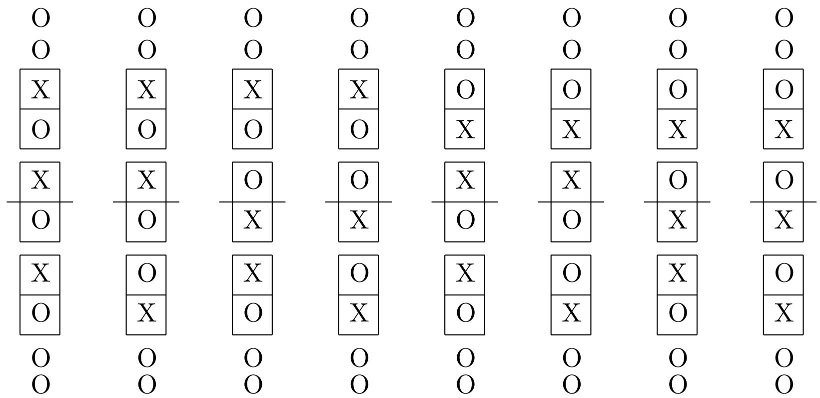

Figure 3: The $2^3 = 8$ (optimal) linear extensions of $\mathbf{3} + \mathbf{7}$ that are balanced around the hub and satisfy the domino condition.

orientations for each domino, thus 2^r different linear extensions are optimal with respect to total linear discrepancy. $\square$

Figure 3 shows all the linear extensions of $\mathbf{3} + \mathbf{7}$ that are optimal with respect to total linear discrepancy.

3 Concluding Remark

We can think of a poset $P = \mathbf{r} + \mathbf{s}$ as representing a line of r people (or projects) and a separate line of s people (or projects) waiting to be served (or completed). A linear extension gives a way of merging these into one line while preserving the order of each line. If $r = s$ or $r = s - 1$ a natural way to accomplish the merge is to have points from each line alternate. Indeed Theorem 13, shows that this is an optimal solution with respect to total linear discrepancy and the other optimal solutions (in the case $r = s$) are not too different. In contrast, the optimal solutions with respect to linear discrepancy insert one line into the middle of the other [8]. In which case, two formerly adjacent members from one line will find themselves separated by the entire length of the second, a situation that might appear unfair to the member of the first line occupying the lower position. By contrast, any optimal solution to the total discrepancy problem never separates two formerly adjacent members by more than a two positions and there exists an optimal solution for which the separation is never more than one.

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