ASYMPTOTICS AND FUNCTIONS

Note: This handout summarizes highlights of CLR Chapter 2. See the book for more details.

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Motivation
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Fine-grained bean counting exposes too much detail for comparing functions.

Want a course-grained way to compare functions that ignores constant factors and focuses on their relative growth in the limit as input sizes get large.

For example, consider:

<table>
<thead>
<tr>
<th></th>
<th>n = 1</th>
<th>n = 1,000</th>
<th>n = 1,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(n)</td>
<td>100n + 1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>q(n)</td>
<td>3n^2 + 2n + 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(n)</td>
<td>0.1n^2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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How Do Your Functions Grow?
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Asymptotic notation is a way of characterizing functions that facilitates comparing their growth in the limit of large inputs. Here is an informal summary of the notation:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Pronunciation</th>
<th>Loosely</th>
</tr>
</thead>
<tbody>
<tr>
<td>f ∈ ω(g)</td>
<td>f is way bigger than g</td>
<td>f &gt; g</td>
</tr>
<tr>
<td>f ∈ Ω(g)</td>
<td>f is at least as big as g</td>
<td>f ≥ g</td>
</tr>
<tr>
<td>f ∈ Θ(g)</td>
<td>f is about the same as g</td>
<td>f = g</td>
</tr>
<tr>
<td>f ∈ O(g)</td>
<td>f is at most as big as g</td>
<td>f ≤ g</td>
</tr>
<tr>
<td>f ∈ o(g)</td>
<td>f is way smaller than g</td>
<td>f &lt; g</td>
</tr>
</tbody>
</table>

Notes:
- Each of ω(g), Ω(g), Θ(g), O(g), o(g) denotes a set of functions. Thus, ω(g) is the set of all functions way bigger than g, Ω(g) is the set of all functions at least as big as g, etc.
- The notation f = ω(g) is really shorthand for f ∈ ω(g).
- The phrases “is at least O(...)” and “is at most Ω(...)” are non-sensical. “Is at least” should be written Ω, and “is at most” should be written O.
Relating the Notations

Here are some of the relationships between the notations:

If \( f \in \omega(g) \), then \( f \in \Omega(g) \).
If \( f \in o(g) \), then \( f \in O(g) \).
\( \Omega(g) \supset \omega(g) \cup \Theta(g) \)
\( O(g) \supset o(g) \cup \Theta(g) \)
\( \Theta(g) = \Omega(g) \cap O(g) \)
\( f \in \omega(g) \) if and only if \( g \in o(f) \)
\( f \in \Omega(g) \) if and only if \( g \in O(f) \)
\( f \in \Theta(g) \) if and only if \( g \in \Theta(f) \)

Warning: unlike numbers, not every pair of functions is comparable!

The following diagram depicts some of these relationships:
Formalizing the Non-tight Bounds ($o$ and $\omega$)

\[ f \in o(g) \text{ if } \lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = 0 \]

\[ f \in \omega(g) \text{ if } \lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = \infty \]

Formalizing the Tight Bounds ($O$, $\Omega$, and $\Theta$)

\[ O(g) = \{ f \mid \text{there exist positive constants } c, n_0 \text{ such that} \]
\[ 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0 \} \]

Think of this as a game. Suppose you claim that $f \in O(g)$. Then you select $c$ and $n_0$, but I try to find a particular $n$ that defeats your claim.

\[ \Omega(g) = \{ f \mid \text{there exist positive constants } c, n_0 \text{ such that} \]
\[ 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0 \} \]

\[ \Theta(g) = \{ f \mid \text{there exist positive constants } c_1, c_2, n_0 \text{ such that} \]
\[ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0 \} \]
Exponentials

Notation:
• $a^n$ = the product of $n$ copies of $a$.
• $a^{-n} = \frac{1}{a^n}$

Key Identities:
• $a^m a^n = a^{m+n}$ \{Special case: $a^0 = 1$.\}
• $(a^m)^n = a^{mn} = (a^n)^m$

Examples:
$$(5^2)^3 =$$
$5^2 \cdot 5^3 =$
$$5^2 \cdot 5^3 =$$
$$25^3 =$$

Logarithms

Notation:
• $\log_b a$ = the power to which $b$ must be raised to equal $a$. (More loosely, it is the number of times that $a$ can be divided by $b$ to reach 1.)
• $\log b ! = -\log Error! a$
• $\lg n = \log_2 n$
• $\ln n = \log_e n$
• $\lg^k(n) = (\lg n)^k$

Key Identities (duals of exponential identities):
• $\log_c(ab) = \log_c(a) + \log_c(b)$ \{Special case: $\log_c 1 = 0$\}
• $\log_c(a^n) = n \cdot \log_c a$

Examples:
$\lg(2n^3) =$
$\ln(32) =$
Relating Exponentials and Logarithms

Key Identities:
• \( b^{(\log_b a)} = a = \log_b(b^a) \)

Examples:
\[
\text{lg Error!} = \\
32^{(\text{lg } n)} =
\]

Asymptotics Involving Exponentials and Logarithms

How do \( \log_2 n \) and \( \log_3 n \) compare?

How do \( 2^n \) and \( 3^n \) compare?

Fact 1: if \( a > 0 \), \( \lim_{n \to \infty} \frac{a^n}{n^b} = \infty \)

Fact 1 implies \( a^n \in \omega(n^b) \).
In other words: Any positive exponential grows faster than any polynomial.

Substituting \( \text{lg } n \) for \( n \) and \( 2^a \) for \( a \) in Fact 1 yields:

Fact 2: if \( a > 0 \), \( \lim_{n \to \infty} \frac{n^a}{\text{lg }^b n} = \infty \)

Fact 2 implies \( n^a \in \omega(\text{lg }^b n) \).
In other words: Any positive polynomial grows faster than any polylogarithmic function.

Factorials

Definition: \( n! = 1 \cdot 2 \cdot 3 \cdots n \)

Stirling’s approximation: \( n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \)

Asymptotics derivable from Stirling’s approximation:
• \( n! = o(n^n) \)
• \( n! = \omega(2^n) \)
• \( \lg(n!) = \Theta(n \lg n) \)