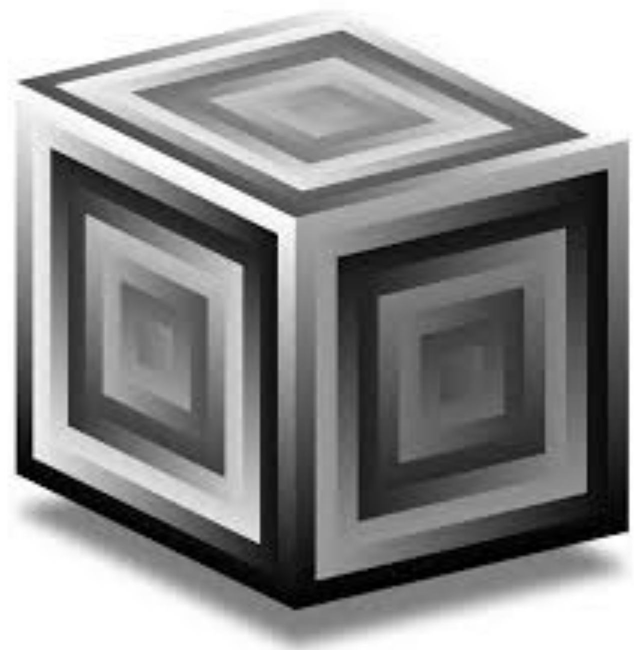
# Frequency Domain

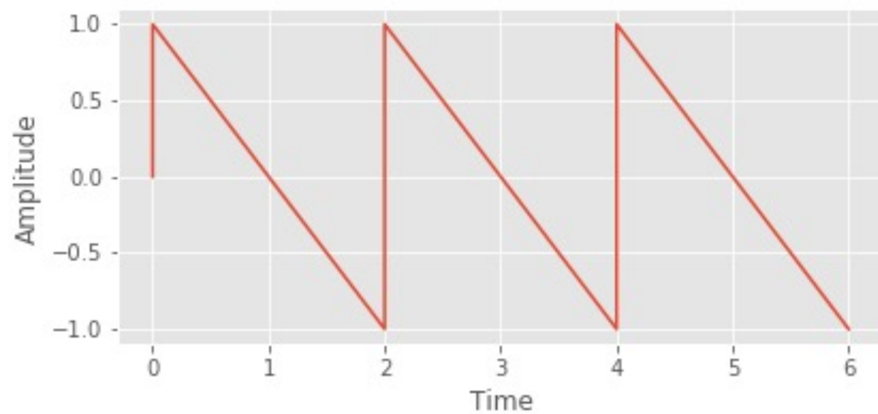
# Topics Addressed

- Frequency Domain
- Orthogonality of Sinusoids
- Periodic Signals and Harmonics
- Inner Product with Sine and Cosine
- Discrete Signals
- Discrete Fourier Transform



# What is the Frequency Domain?

So far we have consider how signals change through time. We call this the time domain. All of our analyses focus on the change in amplitude over time.



# What is the Frequency Domain?

- The frequency domain is the analysis of signals with respect to their constituent frequencies instead of time.

SAWTOOTH WAVE

$$\sum_{n=1}^{\infty} \frac{A}{n} \sin(2\pi f n t)$$

- A signal can be converted to and from the frequency domain using a transform function.
  - There are several flavors of transform functions all of which are named after Joseph Fourier.
  - Remember Fourier stated that any[1] signal can be converted to a sum of sine and cosine waves.

[1]There are some exceptions. See readings.

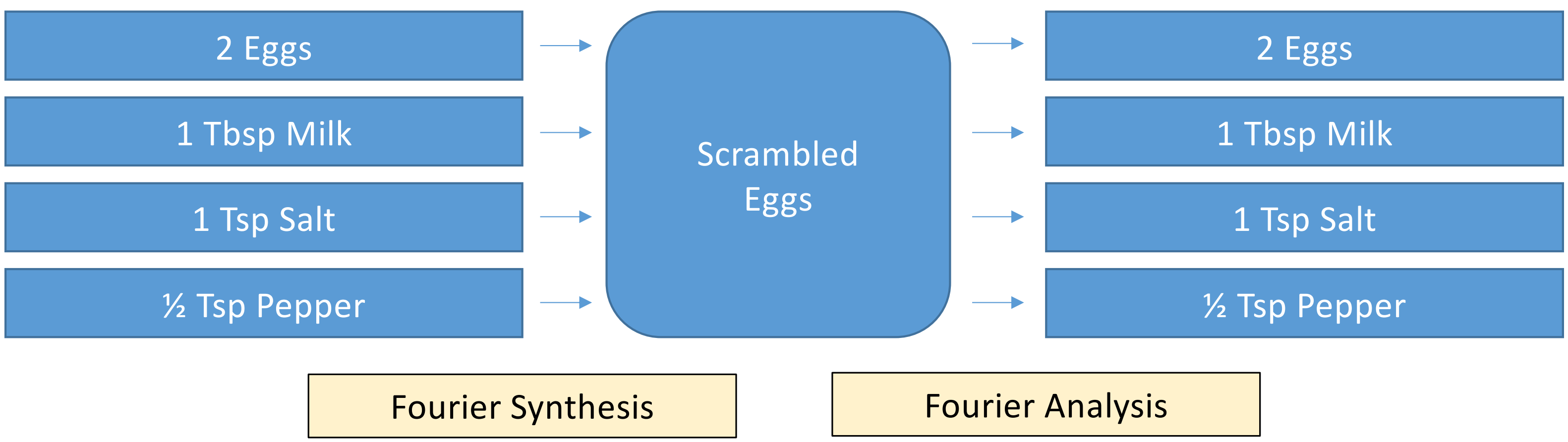
# Terminology

- **Fourier Series**: Expresses a continuous, periodic signal as a summation of weighted harmonics (i.e., sine and cosine waves) from a harmonic series.
- **Fourier Transform:** Expresses any continuous signal as a summation of weighted sinusoids.
- **Discrete Fourier Transform:** Expresses a discrete signal (assumed to be periodic) as a summation of weighted harmonics from a harmonic series.
- **Fourier Analysis:** The process of deconstructing a signal into its constituent sine and cosine waves.
- **Fourier Synthesis:** The process of reconstructing a signal from its constituent sine and cosine waves.

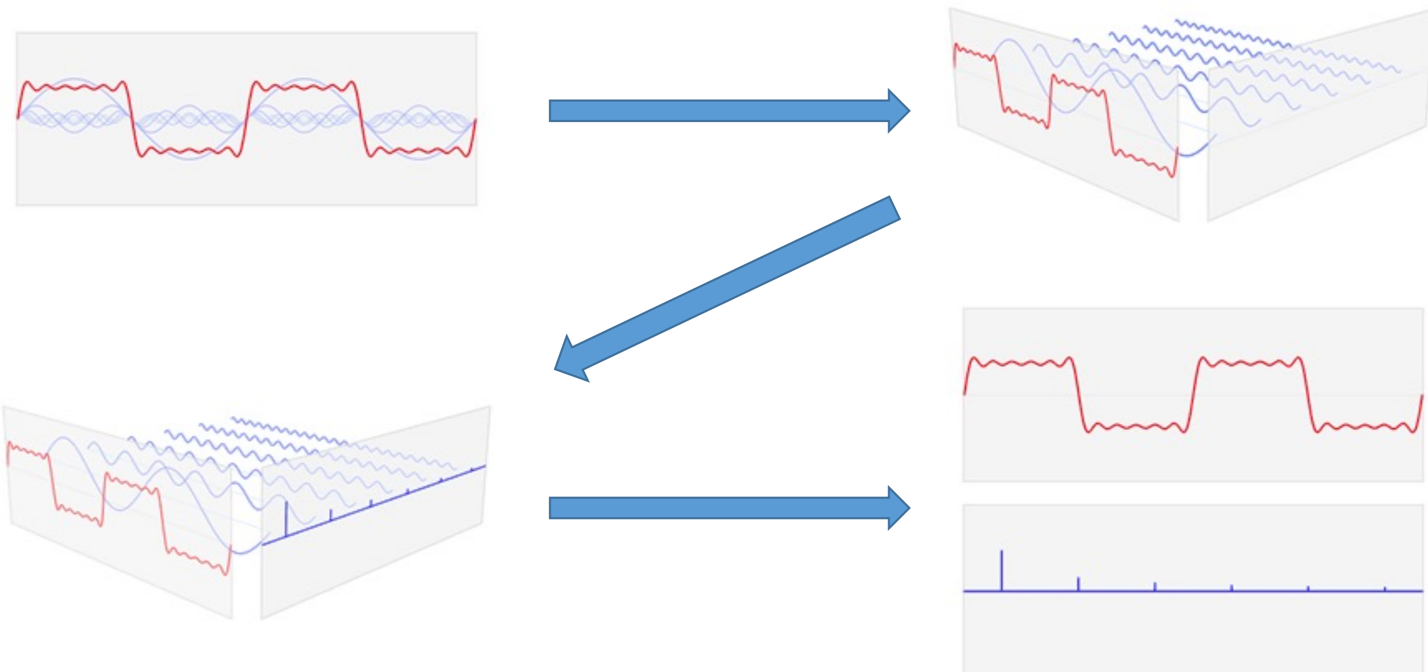
Sadly, we don’t have time to examine the math behind the Fourier Series or Fourier transform but see the Supplemental readings for a more complete understanding.

# Intuition

- We have already been introduced to the notion that complex signals can be created from the basic building blocks of sine and cosine waves: square waves, sawtooth waves, triangle waves, etc.
- A common analogy for Fourier analysis and synthesis is cooking.



# Visualization of a Square Wave

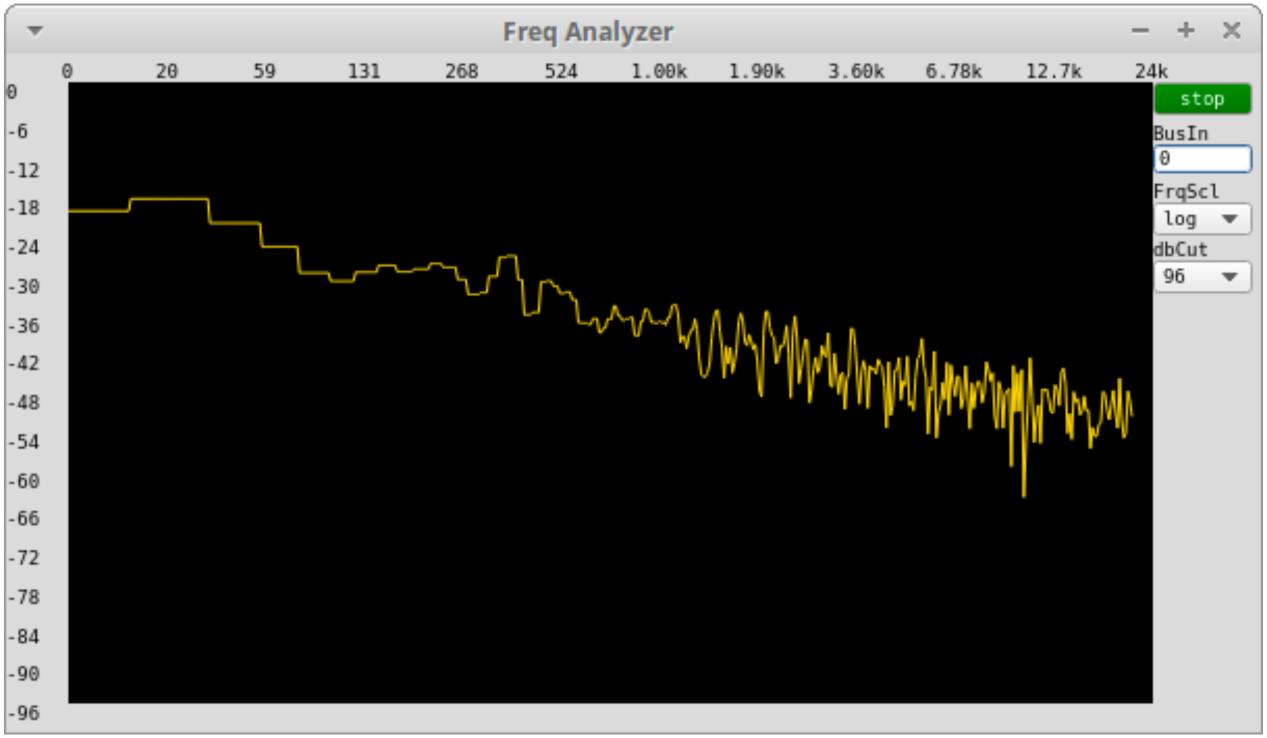


# Types of Fourier Transforms

- Remember our signals are not continuous! They are a construction of samples that are played back at a sample rate and processed through a DAC to render our audio signal.
- If we want to perform a Fourier analysis on an arbitrary musical signal, then we will need to use a Fourier transform that is appropriate for discrete values.
  - Answer: Discrete Fourier Transform! Also known as DFT.
- It turns out that much of the math for signal analysis is computationally intensive. In 1965 by James Cooley and John Tukey developed what is now known as the Fast Fourier Transform which greatly increased the speed of calculations using a divide-and-conquer algorithm that recursively broke down the computations of DFT.
  - DFT algorithms -> $\mathrm{O}(N^2)$
  - FFT algorithms -> $\mathrm{O}(N \log N)$

# FreqScope

The class `FreqScope` that you have been using throughout the semester performs a visualization of a Fourier Analysis on the outputted audio signal.



# GOAL: Derive Intuition Behind How DFT Works

- We are going to start out our discussion of frequency domain by first considering how we can get a signal from the time domain to the frequency domain.
- GOAL: develop intuition about how one would do that.
- Proofs and formalities will be left to class notes.
- DFT requires a lot of math and is complicated.  We will not be too concerned with the details.  Just get a good overview of how the system works.

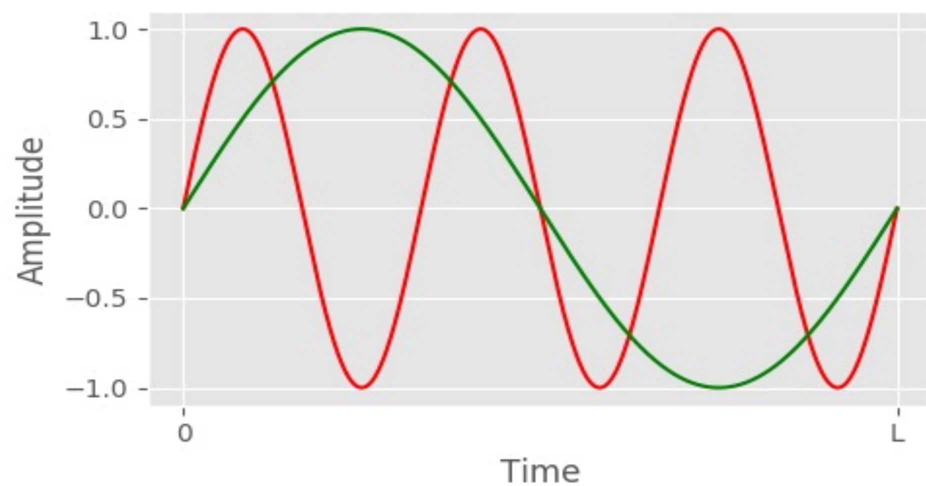
Let’s assume we are dealing with periodic signals…

# Important Idea #1: Sinusoids are Orthogonal

- What is orthogonality?
  - Geometrically, two vectors that are perpendicular
  - Mathematically, dot product = 0
    - What is a dot product?
      - Suppose I have two sampled signals $x = [x_0, x_1, x_2, \dots]$ and $y = [y_0, y_1, y_2, \dots]$ then the dot product of those signals is equal to $x_0 y_0 + \ x_1 y_1 + \ x_2 y_2 + \dots$ etc.
      - For example, if $x = [0, 0.1, 0.15, -0.1]$ and $y = [-1, -0.9, -0.4, 0]$, the dot product of $x$ and $y$ (sometimes written $\langle x, y \rangle$) is $0 * -1 + 0.1 \ * -0.9 + 0.15 \ * -0.4 + -0.1 \ * 0 = -0.15$
    - What about dot product for functions/signals that are not discrete?
      - Technically, this is called the inner product. Dot product is for vectors of finite dimensions.
      - If $f(x)$ and $g(x)$ are functions, then $\langle f, g \rangle = \int_a^b f(x) g(x) dx$. This is a mathematical way of saying to take the multiplicative sum of every infinitesimal point along these two functions.
      - Interested in when $\langle f, g \rangle = 0$

# Important Idea #1: Sinusoids are Orthogonal

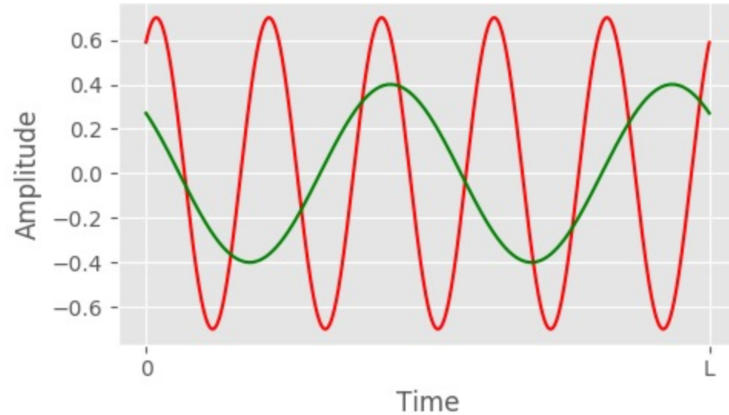
- **CLAIM:** any two sinusoids that are periodic on an interval from 0 to $L$ seconds have an inner product of zero UNLESS they have the same frequency (with one exception).
- Wait what…?

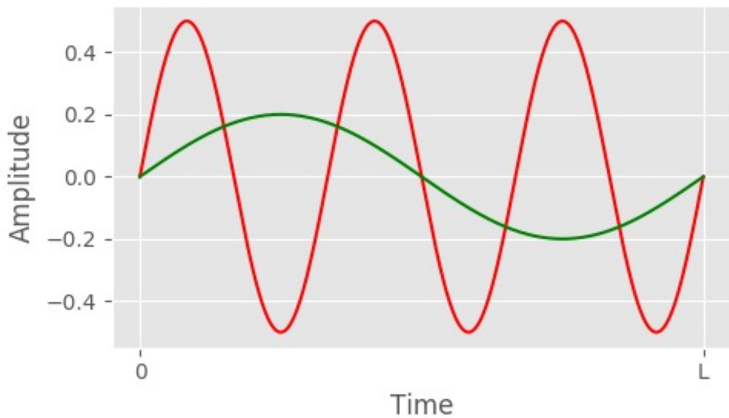


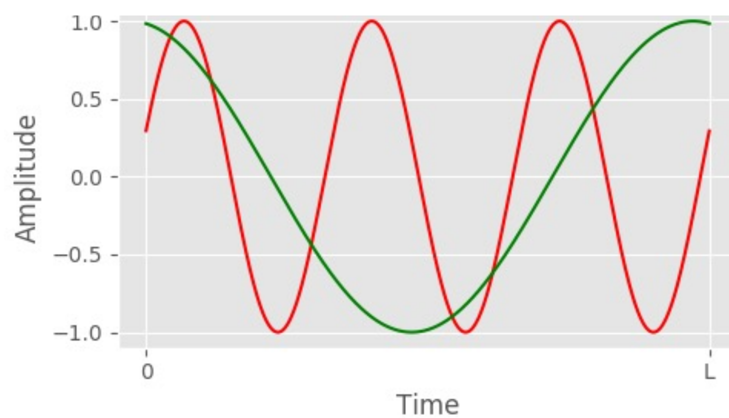

Inner Product of Zero!

# Important Idea #1: Sinusoids are Orthogonal

All of these waves are periodic along the interval 0 to $L$ with different amplitudes or phases and **different frequencies**. All have an inner product of **zero**.

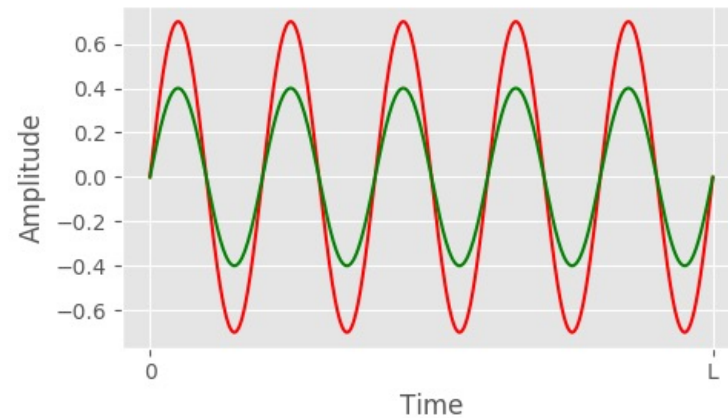


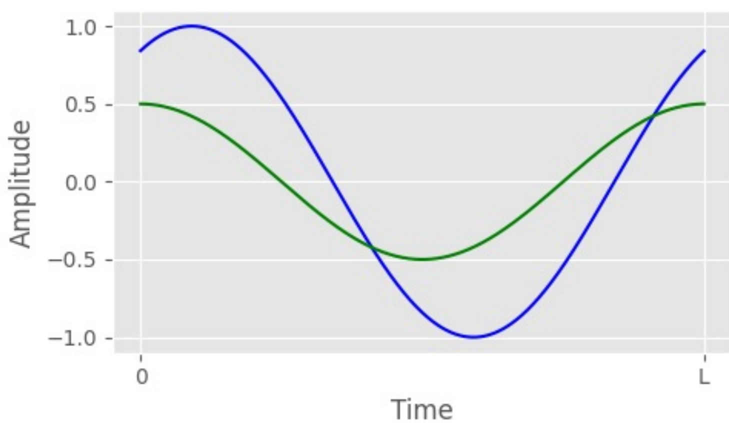



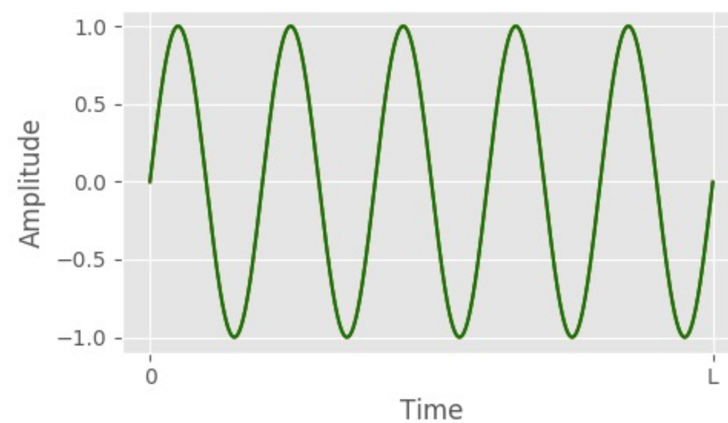



# Important Idea #1: Sinusoids are Orthogonal

All of these waves are periodic along the interval 0 to $L$ with different amplitudes or phases but the **same frequencies**. All have an inner product that is **non-zero**.

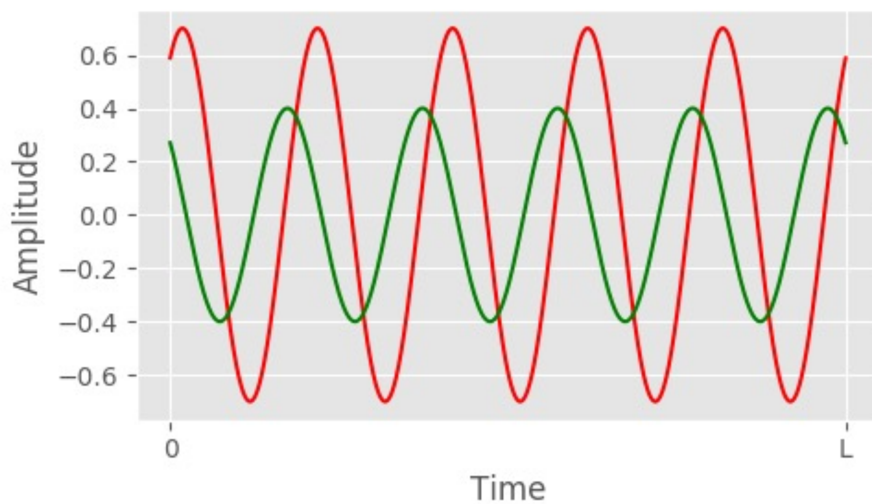


# Important Idea #1: Sinusoids are Orthogonal

ONE EXCEPTION: two sinusoids of the **same frequency** but $\pi/2$ out of phase will also have inner product of 0. For example, the inner product of a sine wave with a cosine wave is always 0 whether they have the same frequency or not.




0! Because out of phase by $\frac{\pi}{2}$.

# Important Idea #1: Sinusoids are Orthogonal

- How can I prove this?
  - This lecture has lecture notes made by me! I have proven everything that we have stated here. Note that you need to have a good grasp on integrals and calculus.
  - But also… you do not need to know the proofs.
- So you have proven it to me, but I still don’t believe you.
  - I wrote a python program that calculates the inner product of two periodic sampled waveforms. Try it out and verify!
- Sampling from the waveforms I described earlier will also lead to similar results.

# Exercises: Orthogonality

State whether the value of the inner product of $f$ and $g$ is zero or non-zero. Note that the inner product is distributive.

| $f$ | $g$ | Zero or non-zero? |
|---|---|---|
| $2\sin(200\pi t + 0.1)$ | $\sin(200\pi t)$ | |
| $\sin(200\pi t)$ | $\sin(400\pi t)$ | |
| $\sin(200\pi t + \frac{\pi}{2})$ | $\sin(200\pi t)$ | |
| $\sin(200\pi t)$ | $\cos(200\pi t)$ | |
| $\sin(200\pi t)$ | $\sin(100\pi t) + \sin(300\pi t)$ | |
| $\sin(400\pi t)$ | $\sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi(100)nt)$ | |

# Important Idea #2: Periodic signals are made of **JUST** harmonics

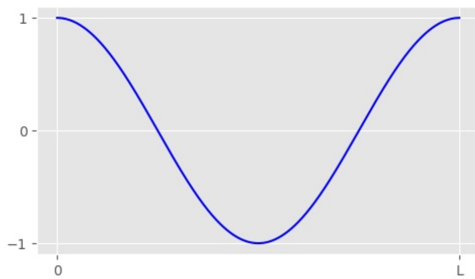
- Joseph Fourier stated that any periodic signal could be reduced to a sum of sine and cosine waves.
- Even crazier, he said that those sine and cosine waves would be integer harmonics of some fundamental. The harmonic series!
  - The amplitude/phase of those harmonics determines the type of periodic signal.
- Fourier series for a **continuous, periodic** signal $f(t)$ over a period of $L$ seconds:

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi \frac{n}{L} t + \phi_n)$$

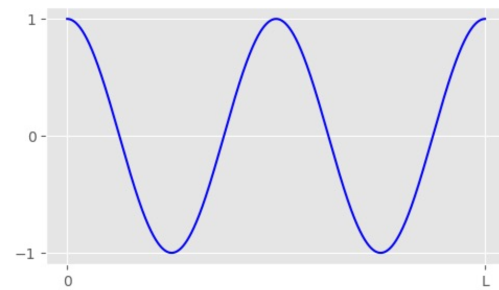
$\frac{n}{L}$ is the frequency of each harmonic. $\frac{1}{L}$ is the frequency of the fundamental of the harmonic series.

# Important Idea #2: Periodic signals are made of **JUST** harmonics

OFFSET




1st Harmonic




2nd Harmonic

…

- Here we are taking the weighted sum of every frequency that could be be periodic along the interval 0 to $L$. These will be frequencies that are integer multiples of the interval $L$. Note that these harmonics could have ANY phase or amplitude.
- Important to remember: some of the harmonics (including the fundamental!) may not be present in the signal itself. Think triangle wave or square wave which only has odd harmonics.

# Exercises: Possible Frequencies

- Suppose a periodic signal has a period of 0.5 seconds. What are the possible frequencies/partials that could constitute that signal?

- Could a frequency of 3Hz be a part of the signal?

# Drawing Some Conclusions…

- How does this help us?
- Remember Fourier stated that any (well-behaved) *periodic* function can be represented as a sum of sinusoids.

Your arbitrary periodic signal as a black box

What's inside?

$$
\begin{gathered}
A_0 \\
A_1\cos(w_1 t + \phi_1) + \\
A_2\cos(w_2 t + \phi_2) + \\
A_3\cos(w_3 t + \phi_3) + \\
\dots
\end{gathered}
$$

# Drawing Some Conclusions…

- IDEA: To figure out if a frequency $f$ is part of an arbitrary periodic, signal $x$, take the inner product of the signal $x$ with a sinusoid of frequency $f$.
  - IF $f$ **IS** a frequency in $x$, then the inner product will be **non-zero**
  - IF $f$ **IS NOT** a frequency in $x$, then the inner product will be **zero**

- If we know the period of the signal (easy to figure out), then we only need to test harmonics!

## Example: $x(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$

- Suppose $L = 1$ (an interval of one second) for simplicity's sake and $x(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$.
  - Note that both sinusoids are periodic on the interval $L = 1$
- If any periodic signal is composed of harmonic partials (that's what the Fourier series tells us), then what are the possible frequencies that could be partials in an arbitrary periodic signal with $L = 1$?

$1,\ 2,\ 3,\ 4,\ 5,$ … etc. Remember that $f = n/L$ in Fourier's Series. Since $L = 1$ and $n$ must be positive integers, then we have the set of positive integers as the possible frequencies.

Example: $x(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$

- How can we test if frequency 1Hz is a part of our signal $x(t)$, still assuming periodicity on the interval of 0 to $L = 1$? Answer: Take the inner product of $x(t)$ with a sinusoid of frequency 1Hz.
- Say $g(t) = \sin(2\pi(1)t)$. What is $\langle x, g\rangle$?
  - Well… it turns out the inner product is distributive! If $a(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3})$ and $b(t) = 0.1\sin(2\pi(4)t + \frac{\pi}{2})$, then $\langle x, g\rangle$ = $\langle a, g\rangle + \langle b, g\rangle$.
  - So what is $\langle a, g\rangle + \langle b, g\rangle$?

0! Because 1Hz is a different frequency then 3Hz and 4Hz.
Remember sinusoids of different frequencies are orthogonal!

## Example: $x(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$

- So we take the inner product with $x$ and sinusoids of frequencies 1Hz and 2Hz and get back inner products of zero both times. Why? Orthogonality of sinusoids.
- What will happen though for $\langle x, \sin(2\pi(3)t)\rangle$?

We will get back a nonzero result! What does this signify? That the frequency 3Hz is a part of $x(t)$. We have sleuthed out one of the partials of our signal. We will learn shortly what exactly that non-zero value can do for us, but for now it simply means we have found a frequency in our signal.

Example: $x(t) = 0.25\sin(2\pi(3)t\ + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$

- Now we decide to take the inner product of $x$ with $\sin(2\pi(4)t)$. What value do we get?

BE CAREFUL! This will lead to a value of zero. Why is that? Remember inner product of two sinusoids that are periodic on the interval $L$ is non-zero when the frequencies are the same and they are NOT out of phase by $\pi/2$.

# To Summarize…

How can I check if a frequency $f_{test}$ is a part of an arbitrary periodic signal $x(t)$?

- First make sure $f_{test} = n/L$ where $n$ is a positive integer; if it is not, then it is not part of the signal. If it is, we still need to test.
- To test, take the inner product of a sinusoid of frequency $f_{test}$ with the signal!
- If your result is zero, then that frequency is not contained. Why? The inner product of $\sin(2\pi f_{test} t)$ with each component of $x(t)$ must have produced zero. Remember that any periodic signal can be constructed in the form $A_1\cos(2\pi f_1 t + \phi_1) + A_2\cos(2\pi f_2 t + \phi_2) + A_3\cos(2\pi f_3 t + \phi_3) +$ etc.
  CAVEAT: possible to miss $f_{test}$ if $\frac{\pi}{2}$ out of phase. We will handle this shortly!
- If your result is non-zero, then your frequency is part of $x(t)$.

# What about that pesky $\pi/2$ business?

- We saw that taking the inner product of $x(t)$ with a sinusoid of given frequency $f_{test}$ does NOT definitively tell us whether $f_{test}$ is a partial of $x(t)$. This is concerning…
- How can we fix this problem?
  - Answer: take the inner product with a sine wave of frequency $f_{test}$ and cosine wave of frequency $f_{test}$. Can not be $\pi/2$ out of phase with both!
  - Example: we saw when $x(t) = 0.25\sin(2\pi(3)t + \frac{\pi}{3}) + 0.1\sin(2\pi(4)t + \frac{\pi}{2})$ and $g(t) = \sin(2\pi(4)t)$ that $\langle x, g\rangle = 0$.
  - Now say $h(t) = \cos(2\pi(4)t)$ then $\langle x, h\rangle$ will be non-zero!
  - So it may be a little less efficient to take the inner product of $x(t)$ for both sine and cosine but it guarantees we will not run into the pesky $\pi/2$ issue.

# Important Idea #3: Take Inner Product with Sine and Cosine

- What does the result of the signal's inner product with a sine or cosine wave signify when the result is non-zero?
- Suppose an arbitrary periodic signal $x(t)$ has partial $A\cos(2\pi f t + \phi)$.
  - Then $\langle x, \cos(2\pi f t)\rangle = \frac{AL}{2}\cos(\phi)$
  - Then $\langle x, -\sin(2\pi f t)\rangle = \frac{AL}{2}\sin(\phi)$[1]
  - $L$ here is the period on which $x(t)$ is periodic.
- Using these two values, we can determine the amplitude AND phase of the partial.
  - Say $a = \frac{AL}{2}\cos(\phi)$ and $b = \frac{AL}{2}\sin(\phi)$,
  - Then $A = \frac{2}{L}\sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$

[1]The choice of taking the inner product with $-\sin(2\pi f t)$ might seem strange. Why not $\sin(2\pi f t)$? Well it turns out that both are $\pi/2$ away from $\cos(2\pi f t)$, so either would work and solve our issue. Here we choose $-\sin(2\pi f t)$ because the Discrete Fourier Transform uses $-\sin(2\pi f t)$.

# Example

Suppose some periodic signal $x(t)$ on the interval 0 to $L = 1$ has frequency $f_{test}$ as one of its partials. We find out that $\langle x, \cos(2\pi f_{test} t)\rangle = 0.25$ and $\langle x, -\sin(2\pi f_{test} t)\rangle = 0$. What is the amplitude and phase of the partial?

$$\boxed{\text{Amplitude} = \frac{2}{1}\sqrt{(0.25)^2 + 0^2} = 0.5}$$

$$\boxed{\text{Phase} = \tan^{-1}\left(\frac{0}{0.25}\right) = 0}$$

$$\boxed{\text{Partial} = 0.5\cos(2\pi f_{test} t)}$$

# Important Idea #3: Take Inner Product with Sine and Cosine

- Taking the inner product with both sine and cosine allows us to reconstruct the original partial based on the two scalar results returned by the inner products!
- You will notice that we have to perform two separate inner products for each frequency we are testing. We can actually express this same idea with one inner product if we use a complex number or vector.
  - Suppose we have a function $c(t) = \cos(2\pi f_{test}t) - \sin(2\pi f_{test}t)i$. Then for some periodic signal $x(t)$ on the period 0 to $L$, $\langle x, c \rangle = \frac{AL}{2}\cos(\phi) + \frac{AL}{2}\sin(\phi)\, i$
  - Using complex numbers yields a single value that contains our two inner products. There is nothing "complex" about our result. There are no imaginary numbers, per se, in play here. No part of our signal contains a square root of a negative number. The advantage is that because you cannot combine the real and complex parts of a complex number, we can keep our two inner products in separate parts of one self-contained value. That is why we use them!

# Important Idea #3: Take Inner Product with Sine and Cosine

$$\langle x, \cos(2\pi f t)\rangle \quad\quad \langle x, -\sin(2\pi f t)\rangle$$

$$\downarrow \quad\quad \downarrow$$

$$\frac{AL}{2}\cos(\phi) \quad\quad \frac{AL}{2}\sin(\phi)$$

$$\langle x, \cos(2\pi f t) - \sin(2\pi f t)\, i\rangle$$

$$\downarrow$$

$$\frac{AL}{2}\cos(\phi) + \frac{AL}{2}\sin(\phi)\, i$$

# Important Idea #3: Take Inner Product with Sine and Cosine

- Euler's formula: $e^{ix} = \cos x + i \sin x$ or $e^{-ix} = \cos x - i \sin x$ .
- So we can also perform inner product with $e^{-2\pi f_{test} t i} = \cos(2\pi f_{test} t) - i \sin(2\pi f_{test} t)$
- Another way to express the same thing, but now as an exponential.
  - Exponential form is elegant because we can encapsulate two sinusoids into a single term.
  - Exponentials are generally easier to manipulate and calculate. Analyses of filters and other DSP properties are done with complex exponentials for this reason

# Important Idea #3: Take Inner Product with Sine and Cosine

$\langle x, \cos(2\pi f t)\rangle$ $\langle x, -\sin(2\pi f t)\rangle$

$\frac{AL}{2}\cos(\phi)$ $\frac{AL}{2}\sin(\phi)$

$\langle x, \cos(2\pi f t) - \sin(2\pi f t)\, i\rangle$

$\frac{AL}{2}\cos(\phi) + \frac{AL}{2}\sin(\phi)\, i$

$\left\langle x, e^{-2\pi f t i}\right\rangle$

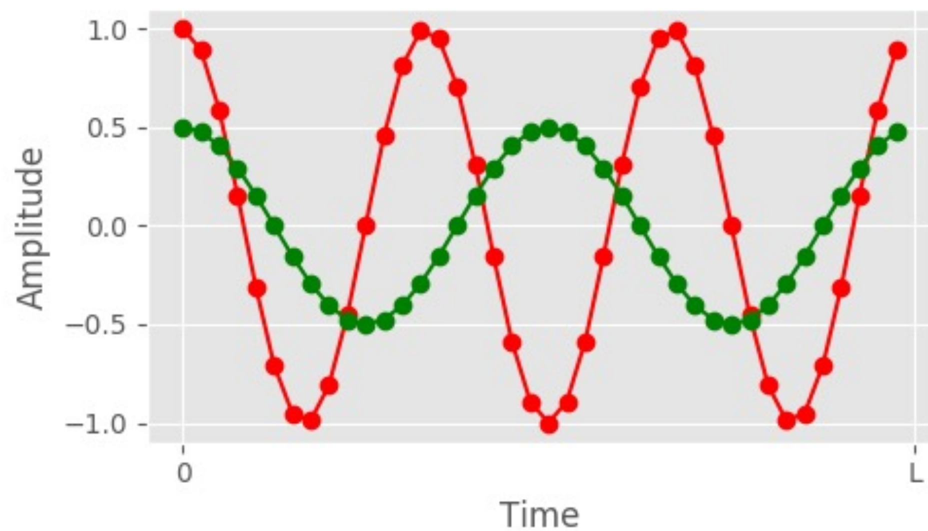
$\frac{AL}{2}\cos(\phi) + \frac{AL}{2}\sin(\phi)\, i$

# Important Idea #4: Discretization

- Everything that we have discussed so far has dealt with continuous signals. If we want to process signals in our computer, we have to understand that those signals are discrete, sampled representations of continuous signals.
- What does the inner product look like in the discrete world? In the continuous world, the inner product is calculated through integrals and calculus. In the discrete world, the calculations are much simpler.
  - If we take the inner product of two discrete signals $f$ and $g$, then we simply sum the product of each sample from $f$ and $g$.
- This of course will be an approximation but all key ideas still apply.

# Calculating Inner Product of Sampled Signals

- Consider two sampled signals $x[n]$ and $y[n]$ where $n$ is $n$th sample of a signal of $N$ total samples.
- The inner product of $x[n]$ and $y[n]$ is equivalent to $\sum_{n=0}^{N-1} x[n]y[n]$.
- The inner product of the two signals on the right will be approximately zero.



# The Discrete Fourier Transform

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-i2\pi kn/N}$$

OR

$$X_k = \sum_{n=0}^{N-1} x[n] \left[\cos\left(2\pi k \frac{n}{N}\right) - i \sin\left(2\pi k \frac{n}{N}\right)\right]$$

This is just an inner product over $N$ samples. Exact same idea!

- $X_k$ is a "frequency bin". It is a complex number whose real and imaginary part contain the information to determine the phase and amplitude of the "frequency" $k$.
- $x$ is the sampled signal as an array of amplitudes. $n$ is the index into the array of samples. $x_n$ is the $n$th sample from the array of amplitudes $x$.
- $k$ is related to the frequency. $k$ is an integer number of complete cycles of a sinusoid over the period $L$ seconds of which we have $N$ samples. Before we were using the variable $n$ to represent this property. We are switching it to $k$, so that we can use $n$ to represent the $n$th sample. Remember samples by themselves do not imply any time period. To derive the frequency we are testing, we can say $f = k/L = (k/N) * f_s$ where $f_s$ is the sample rate. See class notes for derivation.
- $N$ is the number of samples of the signal $x_n$

# Example

- Suppose we want to test whether the discrete periodic signal $x$ has the frequency component $f_{test} = 3Hz$. Let us say that $x$ is periodic on the interval $L = 1$ second and that $x$ has $N = 8$ samples at a sample rate of $f_s = 8Hz$.
- $x = [0.08660254,\ 0.68122488,\ 0.95,\ 0.80369936,\ -0.08660254,$ $-0.68122488,\ -0.95,\ -0.80369936]$.
- We also know then that $k = fL = 3(1) = 3$ for the DFT.
- What are the valid frequencies that could be a part of $x$?

$1Hz, 2Hz, 3Hz$. Fourier Series tells us that $x$ has harmonics that are periodic to $L = 1$. Therefore, $x$ could have $1Hz, 2Hz, 3Hz, 4Hz, \ldots etc$. Presumably though our sampled signal $x$ was passed through a low pass filter to remove any frequencies that could be aliased in our ADC. If we have a sampling rate of 8Hz, then our Nyquist frequency is 4Hz. Thus, we hope our sampled signal $x$ only has frequency components < 4Hz. Therefore $x$ should only contain some combination of $1Hz, 2Hz, 3Hz$.

# Example

- Let us compute $X_k = \sum_{n=0}^{N-1} x[n]\left[\cos\left(2\pi k\,\frac{n}{N}\right) - i\sin\left(2\pi k\,\frac{n}{N}\right)\right] = \sum_{n=0}^{7} x[n]\left[\cos\left(2\pi(3)\,\frac{n}{8}\right) - i\sin\left(2\pi(3)\,\frac{n}{8}\right)\right]$
- Reminder: $x = [0.08660254,\ 0.68122488,\ 0.95,\ 0.80369936,\ -0.08660254,\ -0.68122488,\ -0.95,\ -0.80369936]$.
- Thus, $X_3 = 0.08660254 * (\cos(0) - i\sin(0)) + 0.68122488 * \left(\cos\left(2\pi(3)\,\frac{1}{8}\right) - i\sin\left(2\pi(3)\,\frac{1}{8}\right)\right) + 0.95 * \left(\cos\left(2\pi(3)\,\frac{2}{8}\right) - i\sin\left(2\pi(3)\,\frac{2}{8}\right)\right) \ldots etc = 0.346410161 - 0.2i$
- $X_3 \neq 0$. Therefore, 3Hz is a part of our signal!

# Example

- Let us calculate the phase and amplitude of 3Hz sinusoid.
- We know $\langle x, \cos(2\pi f t)\rangle = \frac{AL}{2}\cos(\phi) = a$ and $\langle x, -\sin(2\pi f t)\rangle = \frac{AL}{2}\sin(\phi) = b$ for a continuous periodic signal. And that $A = \frac{2}{L}\sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$.
- We can use variations of these continuous formulas for discrete signals to determine the amplitude and phase based on our result of $X_3 = 0.346410161 - 0.2i$. Reminder that this will be the phase for a **cosine** wave.
- $\phi = \tan^{-1}(\frac{b}{a}) = \tan^{-1}(\frac{-0.2}{0.346410161}) = -0.5235 \approx -\frac{\pi}{6}$. Same equation in discrete case.
- To get the right approximation in the discrete case, we actually need to calculate $A = \frac{2}{N}\sqrt{a^2 + b^2}$. Therefore, $A = \frac{2}{8}\sqrt{0.346410161^2 + 0.2^2} = 0.1$.
- Therefore, we can conclude that $0.1\cos(2\pi(3)t - \frac{\pi}{6})$ is a part of our signal $x_n$. In fact, the samples come from the following signal $\cos\left(2\pi(1)t - \frac{\pi}{2}\right) + 0.1\cos(2\pi(3)t - \frac{\pi}{6})$.

# Conclusions

- The Discrete Fourier Transform calculates the inner product of a sampled signal with a cosine and sine complex number of a given frequency.
  - The Discrete Fourier Transform will return a complex number whose real and imaginary parts can be used to reconstruct the phase and magnitude of the signal. A non-zero complex number means the frequency is a part of sampled signal.
  - $A = \frac{2}{N}\sqrt{\Re(X_k)^2 + \Im(X_k)^2}$
  - $\phi = \tan^{-1}(\frac{\Im(X_k)}{\Re(X_k)})$
  - $\Re(X_k)$is the real part of the complex number and $\Im(X_k)$is the complex part of the number (i.e., $a + bi$)

# Conclusions

- The Discrete Fourier Transform only returns a complex number for **one** frequency component.
- To get the full picture, we need to perform the DFT for all possible frequencies in the periodic signal. What are the possible frequencies?
  - $N$ (i.e., the number of samples) coupled with the sampling rate determine the time period for the signal. For example, 4 samples at a sampling rate of 8Hz indicates we have a sampled signal of 0.5 seconds. We call this the fundamental period (i.e., the time it would take a sine wave to complete exactly one cycle).
  - We need to test all frequencies that are integer multiples of fundamental period up to but excluding the Nyquist frequency. Equivalent to harmonics of the Fourier Series.
  - Time to perform this operation is $O(n^2)$. FFT does the **exact same** calculations but in $O(n \log n)$ time.

# Conclusions

- The DFT assumes that the sampled signal is periodic.
- Interesting things start happening when we perform the DFT on non-periodic signals. Teaser: spectral leakage!
- The reality of your signal is that it almost certainly will NOT be periodic. We can still use DFT!