# Math Review for CS203

## 1 Sinusoids

What are sinusoids? Sinusoids are sine waves and cosine waves of any periodicity, amplitude and phase. For the purposes of this primer, I will assume that you are familiar with both sine and cosine waves. For reference though, Figure 1 shows graphs of both sinusoids. Note that we are using the variable $t$ here as opposed to something like $x$. Sinusoids in music often relate to changing air pressure as a function of time, hence the variable $t$. We perceive these fluctuations as sound.

### 1.1 Amplitude, Phase, and Frequency

Sinusoids comes in all sorts of variations. The general form of any sine wave is $A\sin(2\pi ft + \phi)$ where $A$ is ampltiude, $f$ is frequency, and $\phi$ is phase. We can express a similar statement for a cosine wave as well: $A\cos(2\pi ft + \phi)$. Let's examine how each of these parameters affect the shape of our sinusoid.

First consider amplitude $A$. Figure 2 shows $\sin(t)$ while the graph on the right shows $2\sin(t)$. What is different about these two graphs? The latter has been stretched vertically by twice the amount. The peak amplitude of $\sin(t)$ is 1 while the peak amplitude of $2\sin(t)$ is 2. This should make intuitive sense. If we multiply the output of $\sin(t)$ by two we should expect twice the output. Mathematically, this is just a specific example of a function transformation. Any function $f(t)$ can be scaled vertically by some factor $k$ by performing $kf(t)$. The larger $k$ is the more vertically stretched the resulting function is. Therefore, the constant $A$ stretches a sinusoid vertically. You'll soon learn that this makes a sound louder.

Let's examine the role $2\pi f$ plays. Figure 3 shows $\sin(t)$ and $\sin(2\pi t)$. What impact did $2\pi$ in $\sin(2\pi t)$ have compared to $\sin(t)$? The period of the sine wave is now smaller. The period of a sine wave is the length of time it takes for the wave to complete one cycle. For $\sin(t)$, the period is $2\pi$ (roughly equivalent to $\approx 6.28$). You can test this yourself. Any value of $\sin(t)$ is equivalent to $\sin(2\pi + t)$. For example, $\sin(0) = \sin(2\pi)$ or $\sin(\frac{\pi}{4}) = \sin(2\pi + \frac{\pi}{4})$. So how does multiplying $t$ by $2\pi$ change the period? Essentially, we have sped up the time it takes for a sine wave to complete its cycle. Take $\sin(0)$. We just stated that $\sin(0) = \sin(2\pi)$. So if we multiply $t$ by $2\pi$, it should take 1 second for the sine wave to complete its cycle. In fact, that is the

Figure 1: A graph of $\sin(t)$ on the left and $\cos(t)$ on the right

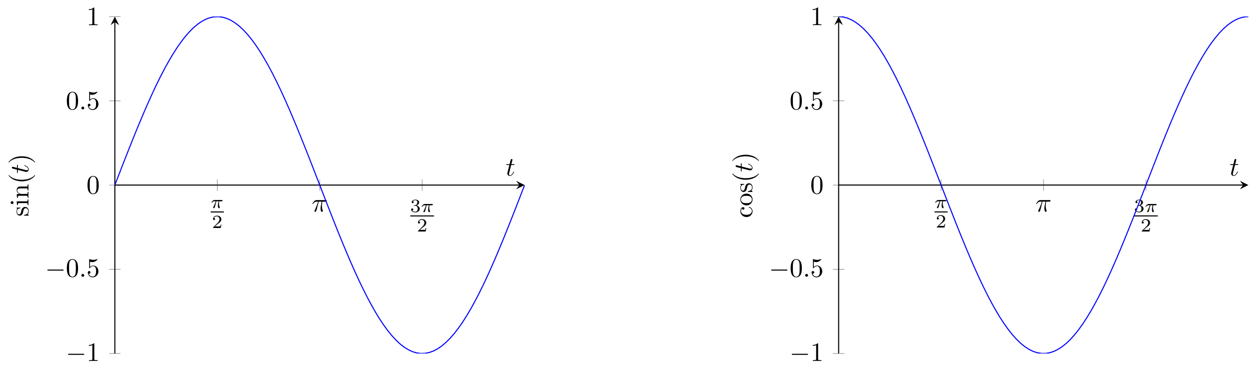


Figure 2: A graph of $\sin(t)$ on the left and $2\sin(t)$ on the right

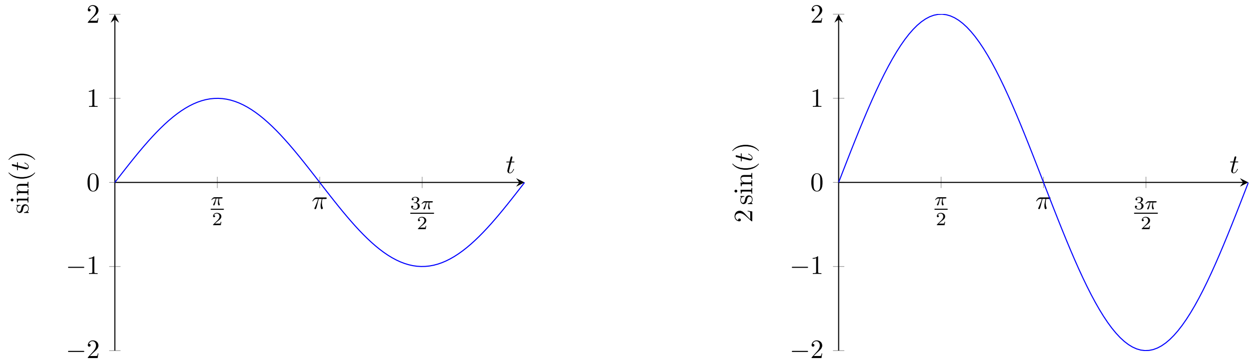
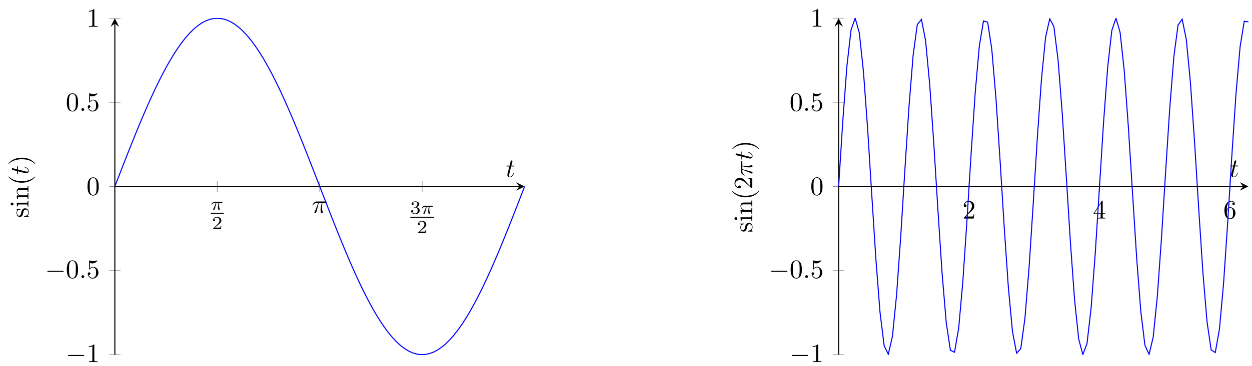



Figure 3: A graph of $\sin(t)$ on the left and $\sin(2\pi t)$ on the right



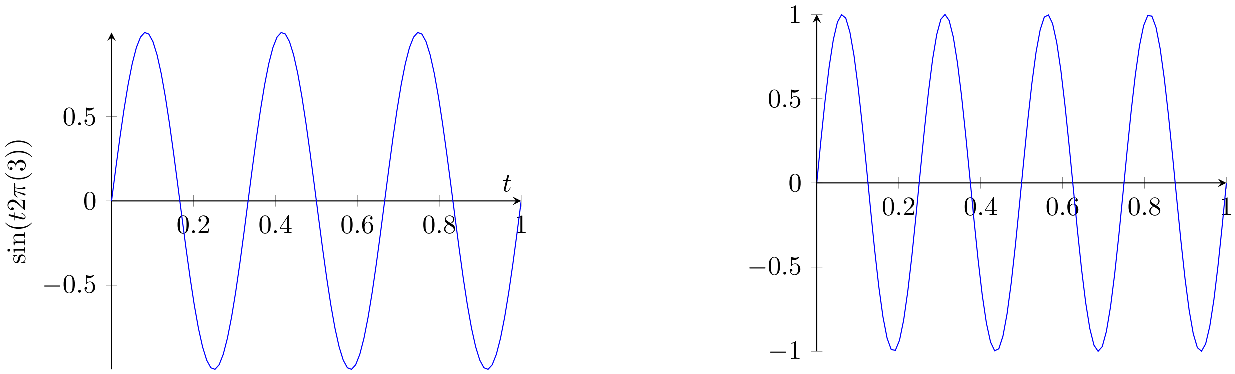

period of $\sin(2\pi t)$! Look again at Figure 3. You'll see that $\sin(2\pi t)$ starts a new cycle on every second.

Like amplitude, multiplying our variable $t$ by some value produces a function transformation. For a function $f(t)$ and constant $k$, $f(kt)$ shrinks or stretches $f(t)$ horizontally. The larger $k$ is the more our function shrinks. When $k$ is a multiple of $2\pi$, we will find that our sine wave completes an integer number of cycles per second. We can generalize this as $2\pi f$ where $f$, the frequency, represents the number of cycles the sine wave completes every second. The unit for cycles per second is Hertz, often abbreviated Hz. The frequency of a sound wave plays a pivotal role in how we perceive the pitch of sound. For this reason, we often want to think about the frequency of our waves. Expressing a sine wave as $\sin(2\pi ft)$ is a nice way of expressing the frequency. As an example, look at Figure 4 which displays $\sin(2\pi(3)t)$ and $\sin(2\pi(4)t)$. Notice that the former completes three cycles in one second and the latter completes four.

Note that you will sometimes see the expression $2\pi f$ in the general form for a sine or cosine wave expressed as $\omega$ as in $A\sin(\omega t + \phi)$ or $A\cos(\omega t + \phi)$. $\omega$ is called the angular frequency. Changes to $\omega$ affect sinusoids in the same way changes to $f$ in $2\pi f$ transform sinusoids.

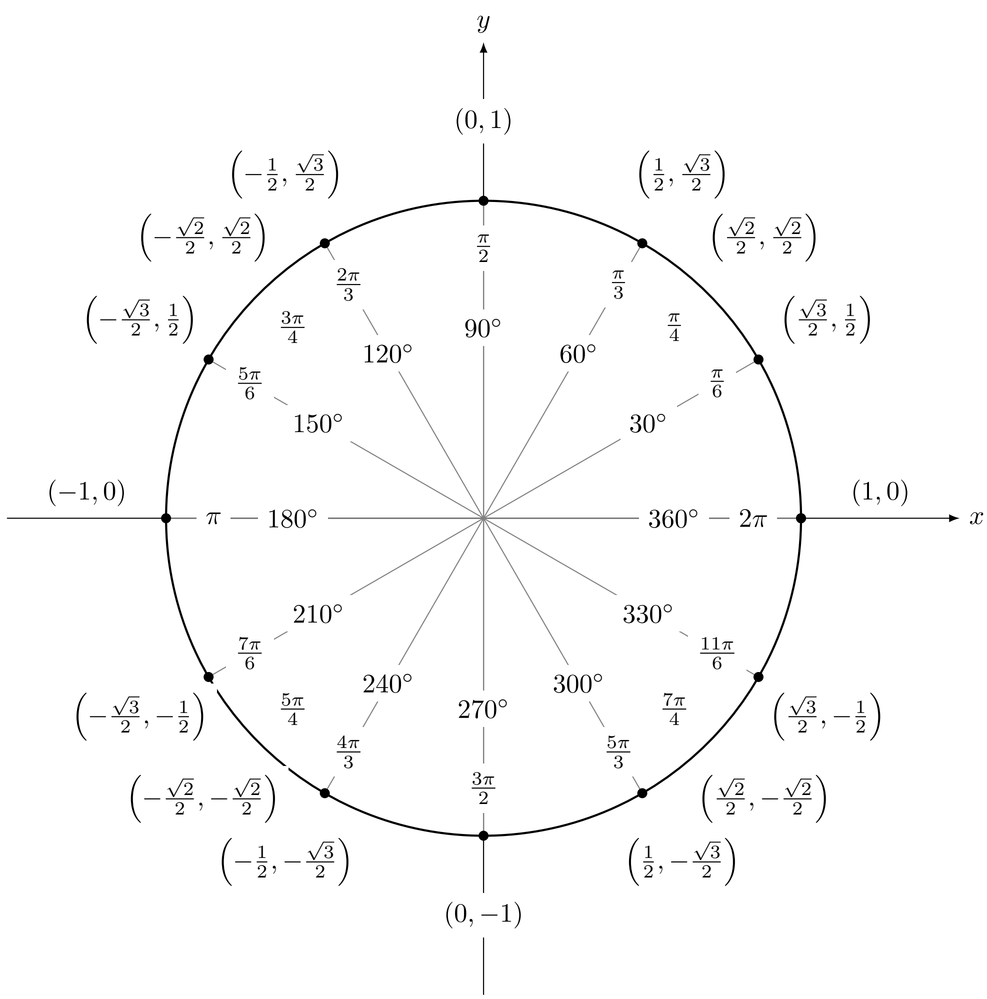
The last parameter to consider is phase, often notated as $\phi$. Look back at Figure 1. If we look at $\sin(t)$ and $\cos(t)$, you will notice they are very similar. If we were to shift $\cos(t)$ to the right by $\frac{\pi}{2}$, it would look exactly like $\sin(t)$. We can say then that $\sin(t)$ and $\cos(t)$ are out of phase by $\frac{\pi}{2}$. Changing $\phi$ shifts a sinusoid to the left or the right. If $f(t)$ is a function and $k$ is a constant, then $f(t+k)$ shifts $f(t)$ to the left or the right by $k$. Note that if $k$ is negative, $f(t)$ is shifted to the right and if $k$ is postive $f(t)$ is shifted to the left. Therefore, we can say then that $\cos(t) = \sin(t + \frac{\pi}{2})$ or that $\sin(t) = \cos(t - \frac{\pi}{2})$. We can't really perceive the phase of a sound. A sine wave of frequency 400Hz sounds exactly the same to us as a sine wave of frequency 400Hz shifted to the right by $\pi/3$. However, phase becomes very important when we start to consider the interaction of multiple sinusoids. Stay tuned!

Figure 4: A graph of $\sin(2\pi(3)t)$ on the left and $\sin(2\pi(4)t)$ on the right




## 1.2 Unit Circle

Interestingly, the output of any sinusoids can be plotted as a circle! The unit circle is a plot that shows the output of $\sin(t)$ and $\cos(t)$. The inner numbers of the circle depict the values of $t$ for both $\sin(t)$ or $\cos(t)$. For the purposes of this class, we will never use degrees. All input values for sinusoids will be radians. The outer values around the circle depict $(\cos(t), \sin(t))$. You should strive to memorize the unit circle as best you can. At the very least, you should be able to recall quickly in which quadrants $\sin(t)$ and $\cos(t)$ are positive/negative.



The unit circle can be very helpful for computing inverse trigonometric functions. For example, consider the function $\sin^{-1}$. What is the result of $\sin^{-1}(-\frac{1}{2})$? An inverse function maps the output of some function (in this case, sin) back to its domain. We can use the unit circle to help solve this problem. Look at the outer numbers along the circle. Where does $\sin(t)$ produce a value of $-\frac{1}{2}$? Answer: at $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$.

We won't use inverse functions too much in this class with the exception of $\tan^{-1}$. Recall that $\tan(t)$ is equivalent to $\frac{\sin(t)}{\cos(t)}$. To compute $\tan^{-1}(x)$, we need to find the ratio of sine to cosine that produces $x$. Consider, $\tan^{-1}(1)$. If we look at the unit circle, we want to find where $\frac{\sin(t)}{\cos(t)} = 1$. There are two locations: at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. If we know that sine and cosine are positive or negative, we can narrow our two possible solutions to one. For example, if sine and cosine were both negative, then we know that the answer must be $\frac{5\pi}{4}$. In general, I will try and indicate positive/negative nature of sine and cosine.

### 1.3 Exercises

1. Graph the function $0.5\sin(2\pi(2)t - \frac{\pi}{2})$ by hand.
2. Consider the difference between $A\sin(2\pi ft + \phi)$ and $-A\sin(2\pi ft + \phi)$.
   (a) Explain how $A\sin(2\pi ft + \phi)$ is transformed when the function is multiplied by $-1$.
   (b) Write an equivalent expression to $-A\sin(2\pi t + \phi)$ that does not use any negative signs. Note the frequency of 1. Hint: consider changing the phase!
3. What is $\tan^{-1}(-\sqrt{3})$ if sin is negative?

## 2 Trigonometric Identities

Trigonometric indentities will be useful throughout the course of this class. In particular, they will be helpful for reducing complex expressions. We will be referring to these identities throughout the class. Not all the identities will be useful to us but for completeness they are listed below.

### 2.1 Identities

You may always use these identities without proof for all exercises in this course.

**Reciprocal Functions**

$$\cot x = \frac{1}{\tan x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$

**Even/Odd**

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

**Pythagorean Identities**

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

**Cofunction Identities**

$$\sin(\frac{\pi}{2} - x) = \cos x$$
$$\cos(\frac{\pi}{2} - x) = \sin x$$
$$\tan(\frac{\pi}{2} - x) = \cot x$$
$$\cot(\frac{\pi}{2} - x) = \tan x$$
$$\sec(\frac{\pi}{2} - x) = \csc x$$
$$\csc(\frac{\pi}{2} - x) = \sec x$$

**Sum and Difference Angles**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

**Double Angles**

$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

**Half Angles**

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

**Power Reduction**

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

**Product To Sum**

$$\sin x \sin y = \frac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]$$
$$\cos x \cos y = \frac{1}{2}\big[\cos(x-y) + \cos(x+y)\big]$$
$$\sin x \cos y = \frac{1}{2}\big[\sin(x+y) + \sin(x-y)\big]$$
$$\tan x \tan y = \frac{\tan x + \tan y}{\cot x + \cot y}$$
$$\tan x \cot y = \frac{\tan x + \cot y}{\cot x + \tan y}$$

## 2.2 Exercises

1. Show that $\csc(\theta)\cos(\theta)\tan(\theta) = 1$.
2. Simplify $\frac{\cot(x)\cos(x)}{\tan(-x)\sin(\frac{\pi}{2}-x)}$.
3. Show that $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ starting from $\frac{\sin(x+y)}{\cos(x+y)}$ and using sum and difference angle identities.

# 3 Summation Notation

Summation notation is a way of succinctly expressing the sum of a series of discrete values or expressions. A very simple example of a sum of dicrete values might be $1 + 2 + 3 + 4 + 5$. This is certainly easy enough to write but consider the sum of all integers from 1 to 1000. That would be quite lengthy to write out. Summation notation provides a shorthand for these kinds of notational issues.

## 3.1 Sigma

Summation notation uses the greek letter $\Sigma$. Consider the simple example again of $1 + 2 + 3 + 4 + 5$. Here is that sum expressed using summation notation.

$$\sum_{i=1}^{5} i$$

Sigma notation defines a variable (in this case $i$) at some starting point below the $\Sigma$. The number above $\Sigma$ represents the end value of $i$ in the series. The expression to the right of $\Sigma$ is the expression to be summed for each value of $i$ starting from $i = 1$ to $i = 5$. You will also see summation notation alternatively expressed as $\sum_{i=1}^{5} i$ where the start and end values of $i$ are to the right of $\Sigma$.

Consider the example below:

$$\sum_{i=1}^{4} i^2$$

What is the example equivalent to? We need to sum the expression $i^2$ for all values of $i$ from 1 to 4. Thus, $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$.

### 3.2 Summation Identities

The following list of identities are meant to be a reference for you throughout the semester. I won't prove them here but most of them should make intuitive sense if you write out a few examples. Note that $c$ here is any constant.

$$\sum_{n=s}^{t} c \cdot f(n) = c \sum_{n=s}^{t} f(n)$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{n=s}^{t} f(n) \pm \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} (f(n) \pm g(n))$$

$$\sum_{n=s}^{t} f(n) = \sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n)$$

### 3.3 Summation Notation and Sinusoids

Why is summation notation relevant to a Computer Music class? You will soon learn that many sounds can be expressed as a sum of sinusoids. It's natural to the way we perceive sound. For example, one of the foundational sounds is a sawtooth wave. A sawtooth wave of frequency $f$ can be expressed as a sum of sinusoids $A\frac{\sin(2\pi(1)ft)}{1} + A\frac{\sin(2\pi(2)ft)}{2} + A\frac{\sin(2\pi(3)ft)}{3} + ...$ etc. It turns out that this is an infinite sum! We can use summation notation to express a sawtooth wave succinctly:

$$\sum_{n=1}^{\infty} \frac{A \sin(2\pi n f t)}{n}$$

### 3.4 Exercises

1. Calculate $\sum_{i=0}^{3}(6 + \sqrt{4^i})$.
2. Write a summation for the following expression: $-\frac{A\sin(2\pi(1)ft)}{1} + \frac{A\sin(2\pi(2)ft)}{4} - \frac{A\sin(2\pi(3)ft)}{9} + \frac{A\sin(2\pi(4)ft)}{16} +$ ...etc. **Hint**: think about how you can use exponents to flip the negative sign.
3. Express the following infinite sum using summation notation: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}....$

# 4 Complex Numbers

You have likely encounter complex numbers prior to college. What exactly are complex numbers? The origin for complex numbers lies in square roots. Recall that the square root of a number $n$ is a number $z$ such that $z^2 = n$. If $n$ is a positive number, there exist exactly two roots: one positive and one negative. For example, say $n = 4$. Then there are two solutions to $z^2 = 4$: $z = 2$ and $z = -2$. Notationally, if we want to express the roots of $n$, we use $\pm\sqrt{n}$ which yields $\pm z$. If we want the positive root of $n$, we simply write $\sqrt{n}$ and if we want the negative root of $n$, we write $-\sqrt{n}$.

The trouble comes when $n = -1$ or any other negative number. What is the solution to $z^2 = -1$? Unfortunately, there does not exist any real number $z$ that satisfies this equation. No number multiplied by itself yields a negative result! Therefore, we say that the solution to this equation is some imaginary number $i$ such that $i^2 = -1$. We know of course that $i$ is not a real number, but providing a framework to reason about such solutions becomes mathematically useful.

Suppose we want to find the solutions to $a^2 = -3$. Now we can do so having defined $i$. First let's start by solving

$$z^2 = 3$$

We can readily see that $z = \pm\sqrt{3}$. Now let's do the following:

$$\begin{aligned} z^2 &= 3 \\ (-1)z^2 &= (-1)3 \\ i^2z^2 &= -3 \\ (zi)^2 &= -3 \end{aligned}$$

Since $z = \pm\sqrt{3}$, then we can say $a = \pm\sqrt{3}i$ for $a^2 = -3$.

It turns out we can think about numbers as having a real and imaginary component. We call them complex numbers. Every number can be written in the form $a + bi$ where $a$ and $b$ are real numbers. $a$ represents the real component of the number and $bi$ represents the imaginary component of the number. All the real numbers that we are familiar with simply have $b = 0$, indicating there is no imaginary component. This implies that the real numbers are a subset of the complex numbers.

There are two basic computational definitions with complex numbers. To add two complex numbers $a + bi$ and $c + di$, we simply compute $(a + b) + (c + d)i$. Notice that the imaginary components add together and the real components add together. To multiply two complex numbers $a + bi$ and $c + di$, we compute $(ac - bd) + (bc + ad)i$.

## 4.1 Complex Numbers and CS203

Why do we care about complex numbers in a computer music class? There is nothing "imaginary" about any musical signal. There are two reasons. We can think of complex numbers as analgous to our $x, y$ coordinate system. The $x$ axis represents the real part of the number and the $y$ axis represents the imaginary part of the number. A point on the coordinate system encodes information about its imaginary and real part separately. As a result, complex numbers provide a convenient system to keep track of two pieces of data. For example, say you want to keep track of apples and oranges, and you have three apples and two oranges. How could we express that using a single number? We could use a complex number. Maybe we say that apples are represented by the real component and oranges are represented by the imaginary component. We might then say that three apples and two oranges is equivalent to $3 + 2i$. There is nothing imaginary about the oranges but we leverage the fact that complex numbers provide a convenient interface to keep track of these two things. This will become important when we discuss the Discrete Fourier Transform.

The second reason is we can relate sinusoids to complex exponentials. Leonhard Euler discovered one of the great equations in mathematics. His equation is presented below:

$$e^{ix} = \cos x + i \sin x$$

The equation brings together some of the most important numbers and functions in math such as sinusoids, $e$, and $i$. Using his identity, we can show the following:

$$\begin{aligned} \cos(\theta) &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin(\theta) &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{aligned}$$

Crazily enough, we can express the sinusoids that we will use in CS203 as complex exponentials! While the above two equations may seem daunting, transforming sinusoids like sin and cos into exponentials can actually make calculations much easier, including proving many of the trigonometric identities stated above. Therefore, a lot of digital signal processing performs calculations on sinusoids in terms of their form in complex exponentials. We will not work much with Euler's identity in this course or complex exponentials but near the end of the course we will need them to understand several foundational ideas in computer music and digital signal processing.

## 4.2 Exercises

1. Find the solutions to $z^2 = -4$.
2. Consider $x = 3 + 2i$ and $y = 2 - i$.
   (a) What is $x + y$?
   (b) What is $xy$?