Lecture 15 – Greedy Algorithms in Graphs

Reading: KT Sections 4.4 and 4.5

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/
Minimum Spanning Tree problem

It’s not about a single shortest path
Repairs on a budget
Repairs on a budget (Formally)

- **Input:**
  - Undirected graph $G = (V, E)$
  - With edge weights $w(u, v)$*

- **Output:**
  - Find a subset of the edges $T \subseteq E$ that connect all the vertices**
  - Such that:
    \[
    w(T) = \sum_{(u, v) \in T} w(u, v)
    \]
    is minimized

* Can weights be negative?
** What are the properties of $T$?
Minimizing the cost
Kruskal in action
Prim in action
Why does this work?

- Simplifying assumption. All edge costs $c_e$ are distinct.

- Cut property. Let $S$ be any subset of nodes, and let $e$ be the min cost edge with exactly one endpoint in $S$. Then the MST contains $e$.

- Cycle property. Let $C$ be any cycle, and let $f$ be the max cost edge belonging to $C$. Then the MST does not contain $f$. 

![Diagram showing examples of cut and cycle properties.](image)
Cycles and Cuts

• Cycle. Set of edges the form a-b, b-c, c-d, ..., y-z, z-a.

• Cutset. A cut is a subset of nodes $S$. The corresponding cutset $D$ is the subset of edges with exactly one endpoint in $S$. 

Cycle $C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1$

Cut $S = \{4, 5, 8\}$
Cutset $D = 5-6, 5-7, 3-4, 3-5, 7-8$
Proof of cut property

• Simplifying assumption. All edge costs $c_e$ are distinct.

• Cut property. Let $S$ be any subset of nodes, and let $e$ be the min cost edge with exactly one endpoint in $S$. Then the MST $T^*$ contains $e$.

• Pf. (exchange argument)
  • Suppose $e$ does not belong to $T^*$, and let's see what happens.
  • Adding $e$ to $T^*$ creates a cycle $C$ in $T^*$.
  • Edge $e$ is both in the cycle $C$ and in the cutset $D$ corresponding to $S$ $\Rightarrow$ there exists another edge, say $f$, that is in both $C$ and $D$.
  • $T' = T^* \cup \{e\} - \{f\}$ is also a spanning tree.
  • Since $c_e < c_f$, cost($T'$) < cost($T^*$).
  • This is a contradiction. $\blacksquare$
Proof of cycle property

- Simplifying assumption. All edge costs $c_e$ are distinct.

- Cycle property. Let $C$ be any cycle in $G$, and let $f$ be the max cost edge belonging to $C$. Then the MST $T^*$ does not contain $f$.

- Pf. (exchange argument)
  - Suppose $f$ belongs to $T^*$, and let's see what happens.
  - Deleting $f$ from $T^*$ creates a cut $S$ in $T^*$.
  - Edge $f$ is both in the cycle $C$ and in the cutset $D$ corresponding to $S \Rightarrow$ there exists another edge, say $e$, that is in both $C$ and $D$.
  - $T' = T^* \cup \{e\} - \{f\}$ is also a spanning tree.
  - Since $c_e < c_f$, cost($T'$) < cost($T^*$).
  - This is a contradiction.  ■