Lecture 5 – Asymptotic Notation
Reading: KT Sections 2.1 and 2.2

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Agenda

2. Algorithm Analysis
   - computational tractability
   - asymptotic order of growth
   - survey of common running times
Algorithm efficiency

• What makes us say that an algorithm is efficient?
  • Real answer: when it’s better than its brute force counter-part

  **Brute force.** For many nontrivial problems, there is a natural brute-force
  search algorithm that checks every possible solution.
  • Typically takes \(2^n\) time or worse for inputs of size \(n\).
  • Unacceptable in practice.

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Polynomial time algorithms

We say that an algorithm is **efficient** if it has a polynomial running time.

**Justification.** It really works in practice!

• In practice, the poly-time algorithms that people develop have low
  constants and low exponents.
• Breaking through the exponential barrier of brute force typically
  exposes some crucial structure of the problem.

**Exceptions.** Some poly-time algorithms do have high constants
and/or exponents, and/or are useless in practice.

**Q.** Which would you prefer \(20n^{1/20}\) vs. \(n^1 + 0.02\ln n\)?
Worst case analysis

**Worst case.** Running time guarantee for any input of size $n$.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

**Exceptions.** Some exponential-time algorithms are used widely in practice because the worst-case instances seem to be rare.

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Other types of analyses

**Worst case.** Running time guarantee for any input of size $n$.

**Ex.** Heapsort requires at most $2 n \log_2 n$ compares to sort $n$ elements.

**Probabilistic.** Expected running time of a randomized algorithm.

**Ex.** The expected number of compares to quicksort $n$ elements is $\sim 2n \ln n$.

**Amortized.** Worst-case running time for any sequence of $n$ operations.

**Ex.** Starting from an empty stack, any sequence of $n$ push and pop operations takes $O(n)$ primitive computational steps using a resizing array.

**Average-case.** Expected running time for a random input of size $n$.

**Ex.** The expected number of character compares performed by 3-way radix quicksort on $n$ uniformly random strings is $\sim 2n \ln n$.

**Also.** Smoothed analysis, competitive analysis, ...
The way things grow

![Graph showing growth of cost with respect to size, illustrating linear, quadratic, exponential, and logarithmic growth]

By the numbers

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$ log $n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$18$ min</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$11$ min</td>
<td>$36$ years</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$1$ sec</td>
<td>$12.892$ years</td>
<td>$10^{17}$ years</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>$&lt; 1$ sec</td>
<td>$1$ sec</td>
<td>$1$ sec</td>
<td>$18$ min</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>$&lt; 1$ sec</td>
<td>$2$ min</td>
<td>$12$ days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>$&lt; 1$ sec</td>
<td>$3$ hours</td>
<td>$32$ years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>$1$ sec</td>
<td>$20$ sec</td>
<td>$12$ days</td>
<td>$31,710$ years</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>

Table 2.1: The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{20}$ years, we simply record the algorithm as taking a very long time.
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Big-Oh notation

**Upper bounds.** $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $T(n) \leq c \cdot f(n)$ for all $n \geq n_0$.

- This means that $T(n)$ grows no faster than $f(n)$.
- For example, let’s consider $17n^2$ and $n^2$.
Common mistakes

Equals sign. \( O(f(n)) \) is a set of functions, but computer scientists often write \( T(n) = O(f(n)) \) instead of \( T(n) \in O(f(n)) \).

Ex. Consider \( f(n) = 5n^3 \) and \( g(n) = 3n^2 \).
- We have \( f(n) = O(n^3) = g(n) \).
- Thus, \( f(n) = g(n) \). \( \times \)

Domain. The domain of \( f(n) \) is typically the natural numbers \( \{0, 1, 2, \ldots\} \).
- Sometimes we restrict to a subset of the natural numbers.
- Other times we extend to the reals.

Non-negative functions. When using big-Oh notation, we assume that the functions involved are (asymptotically) non-negative.

Big-Omega notation

Lower bounds. \( T(n) \) is \( \Omega(f(n)) \) if there exist constants \( c > 0 \) and \( n_0 \geq 0 \) such that \( T(n) \geq c \cdot f(n) \) for all \( n \geq n_0 \).

Typical usage. Any compare-based sorting algorithm requires \( \Omega(n \log n) \) compares in the worst case.

Meaningless statement. Any compare-based sorting algorithm requires at least \( O(n \log n) \) compares in the worst case.
Big-Theta notation

**Tight bounds.** $T(n)$ is $\Theta(f(n))$ if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$ for all $n \geq n_0$.

**Ex.** $T(n) = 32n^2 + 17n + 1$.
- $T(n)$ is $\Theta(n^2)$, \hspace{1cm} \text{choose } c_1 = 32, c_2 = 50, n_0 = 1
- $T(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$.

**Typical usage.** Mergesort makes $\Theta(n \log n)$ compares to sort $n$ elements.

Some properties to know...

**Polynomials.** Let $T(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then, $T(n)$ is $\Theta(n^d)$.

**Pf.** \[ \lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0 \]

**Logarithms.** $\Theta(\log_b n)$ is $\Theta(\log_a n)$ for any constants $a, b > 0$.

**Logarithms and polynomials.** For every $d > 0$, $\log n$ is $O(n^d)$.

**Exponentials and polynomials.** For every $r > 1$ and every $d > 0$, $n^d$ is $O(r^n)$.

**Pf.** \[ \lim_{n \to \infty} \frac{n^d}{r^n} = 0 \]
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Can you think of a?

- Linear time algorithm
- Sublinear time algorithm
- Linearithmic time algorithm
- Quadratic time algorithm
- Cubic time algorithm