Lecture 8 – Graphs

Reading: KT Sections 3.1

Partial content of these slides have been obtained from the official lecture slides that accompany the textbook. A complete set of slides can be found at: http://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Let’s start with some basics

Graph representation
Undirected Graphs

- Undirected graph. $G = (V, E)$
  - $V =$ nodes.
  - $E =$ edges between pairs of nodes.
  - Captures pairwise relationship between objects.
  - Graph size parameters: $n = |V|$, $m = |E|$.

$$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$E = \{1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6\}$$
$n = 8$
$m = 11$

Some Graph Applications

<table>
<thead>
<tr>
<th>Graph</th>
<th>Nodes</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>transportation</td>
<td>street intersections</td>
<td>highways</td>
</tr>
<tr>
<td>communication</td>
<td>computers</td>
<td>fiber optic cables</td>
</tr>
<tr>
<td>World Wide Web</td>
<td>web pages</td>
<td>hyperlinks</td>
</tr>
<tr>
<td>social</td>
<td>people</td>
<td>relationships</td>
</tr>
<tr>
<td>food web</td>
<td>species</td>
<td>predator-prey</td>
</tr>
<tr>
<td>software systems</td>
<td>functions</td>
<td>function calls</td>
</tr>
<tr>
<td>scheduling</td>
<td>tasks</td>
<td>precedence constraints</td>
</tr>
<tr>
<td>circuits</td>
<td>gates</td>
<td>wires</td>
</tr>
</tbody>
</table>
World Wide Web

- Web graph.
  - Node: web page.
  - Edge: hyperlink from one page to another.

9-11 Terrorist Network

- Social network graph.
  - Node: people.
  - Edge: relationship between two people.

Ecological Food Web

- Food web graph.
  - Node = species.
  - Edge = from prey to predator.

Paths and Connectivity

- Def. A path in an undirected graph $G = (V, E)$ is a sequence $P$ of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair $v_i, v_{i+1}$ is joined by an edge in $E$.

- Def. A path is simple if all nodes are distinct.

- Def. An undirected graph is connected if for every pair of nodes $u$ and $v$, there is a path between $u$ and $v$. 
Cycles

- **Def.** A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k-1$ nodes are all distinct.

![Cycle diagram](image)

cycle $C = 1-2-4-5-3-1$

Trees

- **Def.** An undirected graph is a tree if it is connected and does not contain a cycle.

- **Theorem.** Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements imply the third.
  - $G$ is connected.
  - $G$ does not contain a cycle.
  - $G$ has $n-1$ edges.

![Tree diagram](image)
Rooted Trees

- Rooted tree. Given a tree $T$, choose a root node $r$ and orient each edge away from $r$.
- Importance. Models hierarchical structure.

Exercise time!
Graph Representation: Adjacency Matrix

- Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.
  - Two representations of each edge.
  - Space proportional to ________.
  - Checking if (u, v) is an edge takes $\Theta(?)$ time.
  - Identifying all edges takes $\Theta(?)$ time.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
8 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Graph Representation: Adjacency List

- Adjacency list. Node indexed array of lists.
  - Two representations of each edge.
  - Space proportional to ________.
  - Checking if (u, v) is an edge takes $O(?)$ time.
  - Identifying all edges takes $\Theta(?)$ time.