Directed Graphs: Strongly Connected Components, and Topological Sorting

Reading: KT 3.5—3.6

CS231 Fundamental Algorithms
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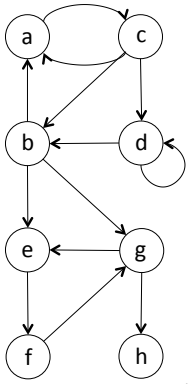
Tue April 12, 2022 *(Revised Thu Apr 28)*

Directed Graphs



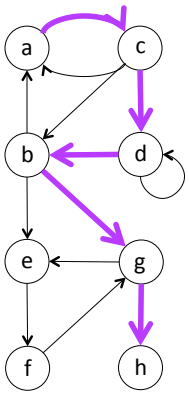
A **directed graph** is a pair *(V, E)* of
1. A set *V* of **vertices** (also called **nodes**)
2. A set *E* of **directed edges** (where each edge is a **pair** of two nodes†)

vertices
({a,b,c,d,e, f, g, h},
{ (a,c), (b.a), (b,e), (b,g), (c,a), (c,b), (c,d),
(d,d), (e,f), (f,g), (g,e), (g,h) })
directed edges

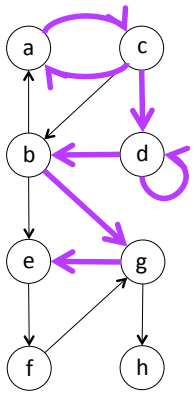
† This definition **does** allow a **self-edge** from a vertex to itself.

Directed Paths

A **path** in a directed graph is a sequence of vertices where each vertex is connected to the next by a directed edge. A path is **simple** if no vertices are repeated. The **length** of the path is one less than the length of the sequence.



simple path (a,c,d,b,g,h); length=5



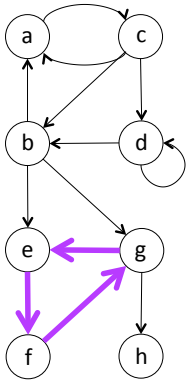
nonsimple path (c,a,c,d,d,b,g,e); length=7

Note: there's a length 0 path from every node to itself. E.g., (a)

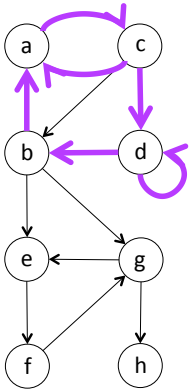
Directed Cycles

A **cycle** in a directed graph is a path with length ≥ 1 beginning and ending at the same vertex.

A **simple cycle** is a cycle that repeats no vertices except the first/last.



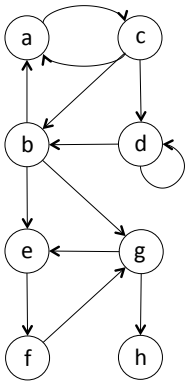
Simple cycle (e,f,g,e)



Nonsimple cycle (a,c,a,c,d,d,b,a)

Mutual Reachability

In a directed graph, vertices *u* and *v* are **mutually reachable** if there's a path from *u* to *v* and from *v* to *u*.



Examples of mutually reachable vertex pairs:
(a,a), (a,b), (a,c), (a,d), (b,c), (b,d), (c,d), (e,f), (e,g), (f,g)

Examples of vertex pairs that aren't mutually reachable:
(a,e), (a,f), (a,g), (a,h), (b,e), (b,f), (b,g), (b,h), (c,e), (c,f), (c,g), (c,h), (d,e), (d,f), (d,g), (d,h), (e,h), (f,h), (g,h)

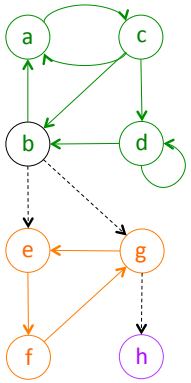
Mutual reachability is:

- **reflexive**: For all vertices *v*, *(v,v)* is mutally reachable.
- **symmetric**: For all vertices *u*, *v*, if *(u,v)* is mutually reachable, then *(v,u)* is mutually reachable.
- **transitive**: For all vertices *u*, *v*, *w*, if *(u,v)* and (v,w) are mutually reachable, then *(u,w)* is mutually reachable.

So mutual reachability is an **equivalence relation**.

Strongly Connected Components

In a directed graph *G*, a **strongly connected component** (or just **strong component**) is a subgraph of *G* consisting of all vertices in the same mutually reachable equivalence class, along with all edges that connect these vertices. Edges not in any strongly connected component are **cross edges** between components.



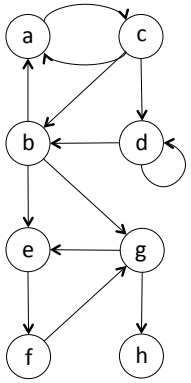
A graph can always be partitioned into a collection of strongly connected components, where each vertex is a member of exactly one such component.

The example graph has 3 strongly connected components

In-degree and Out-degree

The **in-degree** of a vertex is the number edges entering that vertex.

The **out-degree** of a vertex is the number edges exiting that vertex.



in-degree(c) = in-degree(f) = in-degree(h) = 1

in-degree(a) = in-degree(b) = in-degree(d)
= in-degree(e) = in-degree(g) = 2

out-degree(h) = 0

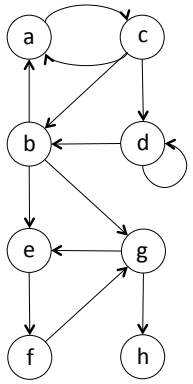
out-degree(a) = out-degree(e) = out-degree(f)
= out-degree(g) = 1

out-degree(d) = 2

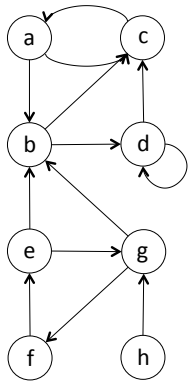
out-degree(b) = out-degree(c) = 3

Directed Graph Reversal

If *G* is a directed graph, its reversal G^{rev} has the same vertices, but all edges in the opposite direction.



Directed graph *G*



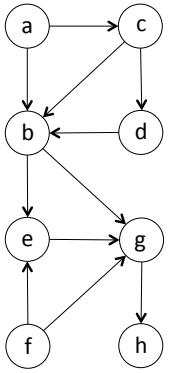
Directed graph G^{rev}

Directed Acyclic Graphs (DAGs)

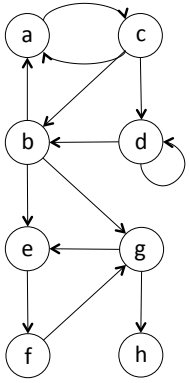
A directed graph *G* is a **directed acyclic graph (DAG)** iff it has no cycles.

In practice, DAGs are used to represent prerequisites/dependencies, and tree-like structures with shared descendants.

This graph *H* is a DAG



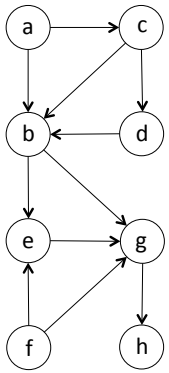
This graph *G* is **not** a DAG



Topological Sort/Ordering on a DAG

A **topological sort** (or **topological order**) of a DAG *D* is a sequence *S* of all vertices from the DAG such that if there's an edge in *D* from S_i to S_j , then $i < j$.

This graph *H* is a DAG



Some topological sorts of H:

(a, c, d, b, f, e, g, h)

(a, c, d, f, b, e, g, h)

(a, c, f, d, b, e, g, h)

(a, f, c, d, b, e, g, h)

(f, a, c, d, b, e, g, h)

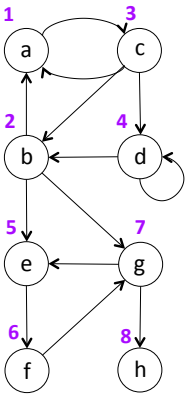
Are there any others?

Directed Graph Representation:

We'll assume the *n* vertices of graph have unique IDs (UIDs) numbered 1 .. *N*, and that vertex names and edges (as adjacency lists for *outgoing edges*) are indexed by UID.

This is like undirected graphs, except that *u* in *OutAdj[v]* does not imply *v* in *OutAdj[u]*.

In some contexts, the directed graph might also have *InAdj* for incoming edges



Name array *Name* with 1-indexed arrays

```
#  1    2    3    4    5    6    7    8
[ 'a', 'b', 'c', 'd', 'e', 'f', 'g', 'h' ]
```

Adjacency array *OutAdj* with 1-indexed arrays. (Adjacent vertices are sorted in example, but need not be)

```
[ [3], # OutAdj[1]
  [2, 5, 7], # OutAdj[2]
  [1, 2, 4], # OutAdj[3]
  [2, 4], # OutAdj[4]
  [6], # OutAdj[5]
  [7], # OutAdj[6]
  [5, 8], # OutAdj[7]
  [] ] # OutAdj[8]
```


Some Typical Directed Graph Questions

What are some typical questions we want to answer about directed graphs?

- Is there a path between two vertices?
- What is the shortest path between two vertices? If edges are annotated with distances/weights, what is the shortest distance or minimal weight path between two vertices?
- Are two vertices mutually reachable? (I.e., are they in the same strongly connected component?)
- What are the strongly connected components of a graph?
- Does the graph have a cycle? Is there a cycle involving a particular vertex or edge?
- Is a graph a DAG?
- If a graph is a DAG, what is a topological sort of its vertices?
- Can we label the vertices/edges of a graph with information such that certain properties hold?

Topological Sort Algorithm

```
function TopSort(D) # If D is a DAG, return an array that's a
                    # sequence of the vertices of D in an order
                    # respecting the directed edges in D.
                    # Otherwise indicate that D has a cycle.
    Remaining <- V(D) # Set of vertices to be processed.
    Seq = new empty array # Initialize result sequence
    while Remaining is nonempty
        choose a v in Remaining where len(InAdj[v]) = 0
            # One must exist or there'd be a cycle!
            # If one does not exist, can return an indication
            # that there's a cycle.
        Seq@v # Extend result sequence with v
        Remove all edges in OutAdj[v] from D
            # affects InAdj for adjacent nodes
    return Seq
```


Another Topological Sort Example