CS235 Languages and Automata — **Handout # 1**
Prof. Lyn Turbak — **September 6, 2007**
Wellesley College

Problem Set 1

Due: At the beginning of class, Thursday, September 13

Required Reading: Stoughton, Ch. 1

Optional Reading: Sipser, Ch. 0

Problem 1 Give two proofs that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:

a. Give a "proof by picture", using Venn diagrams.

b. Give a symbolic proof that does not use any pictures.

Problem 2 Suppose you are given the following two properties:

Prop 1: $\overline{\overline{A}} = A$

Prop 2: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (DeMorgan's law (1) from Slide 2-12).

Give an algebraic proof that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (DeMorgan's law (2) from Slide 2-12).

Problem 3 Suppose F is any finite subset of *Nat* and G is any undirected graph with a finite number of vertices. Consider the following relations:

sameMod5 $= \{(a, b) \mid a, b \in \mathit{Int}$ and $(a - b) \bmod 5 = 0\}$

differsByOneChar $= \{ (r, s) \mid r, s \in \mathit{String}$, length$(r) =$ length$(s) > 0$,
and r and s have the same characters at every index but one.$\}$

connected $= \{(v, w) \mid v, w$ are vertices in G and (v, w) is an edge in $G\}$

pairLEQ $= \{(p, q) \mid p, q \in \mathit{Nat} \times \mathit{Nat}$, first(p) $\leq$ first(q), and second(p) $\leq$ second(q)$\}$

hasOneLessElt $= \{(A, B) \mid A, B \in \mathcal{P}(F)$ and $B = A \cup \{n\}$ for some $n \in F$ s.t $n \notin A\}$

a. For each of the above relations, indicate *all* of the following properties that it has: reflexive, symmetric, transitive. Which of the relations are equivalence relations?

b. For each of the above relations, describe (in English) its (1) reflexive closure; (2) transitive closure; and (3) reflexive, transitive closure.

Problem 4 (from Sipser) Find the error in the following proof that $2 = 1$. Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$, to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

Problem 5 (a Randy classic) Professor Shelby Fuddled wishes to give exactly one A in a class of three very bright students. She chalks her left index finger and asks the three students to close their eyes. Professor Fuddled says that she will touch the forehead of each student with either her left or right index finger while their eyes are closed. In point of fact she touches each student with her left index finger thereby leaving a mark of chalk on each. She then asks them to open their eyes, look around and raise their hand if they see a chalk mark on one or both of the other two students. All three students raise their hands. Fuddled then asks the students to determine (without touching their foreheads, looking in the mirror, or communicating with anyone) whether or not their own heads are marked with chalk. The first student to determine this answer will be awarded an A for the semester. After five minutes, one of the students takes her hand down and announces that her head is chalked. How did she know? *(Hint: Think contradiction.)*

Problem 6 Consider the following sets:

$$Bool = \{T,\ F\}$$
$$Sign = \{-,\ 0,\ +\}$$
$$Nat = \{0,\ 1,\ 2,\ 3,\ \ldots\}$$

For each of the following sets,

- if the set is finite, state its size. (You may want to list all of its elements to check your answer.)
- if the set is countably infinite, prove this fact by defining a bijection betwen Nat and the set.
- if the set is uncountably infinite, prove this fact by using a diagonalization argument.

You needn't give any proof for part **n**, which is already discussed in Stoughton.

a.	$Bool \times Bool$	**f.**	$Bool \to Bool$	**k.**	$Nat \to Bool$
b.	$Bool \times Sign$	**g.**	$Bool \to Sign$	**l.**	$\mathcal{P}(Bool)$
c.	$Sign \times Bool$	**h.**	$Sign \to Bool$	**m.**	$\mathcal{P}(Sign)$
d.	$Sign \times Sign$	**i.**	$Sign \to Sign$	**n.**	$\mathcal{P}(Nat)$
e.	$Bool \times Nat$	**j.**	$Bool \to Nat$		

Problem 7 (from Sipser) Show that every graph with 2 or more nodes contains two nodes that have equal degrees. *(Hint: this is a counting problem. Use the pigeon-hole principle.)*

Problem 8 For any set S, define a bijection between $\mathcal{P}(S)$ (the power set of S) and $S \to Bool$.

Problem 9 Use induction to prove the following fact: the sum of the first k odd natural numbers is k^2.

Problem 10 (adapted from Stoughton) Use strong induction to prove the following fact: for all $n \in Nat$, if $n \geq 8$, then there are $j, k \in Nat$ such that $n = 3j + 5k$.

Problem 11 (adapted from Sipser) Find the error in the following proof that all horses are the same color.

Claim: In any set of n horses, all horses are the same color.

Proof: By induction on n

Basis Step ($n = 1$) In any set containing just one horse, all horses clearly are the same color.

Induction step ($n > 1$). Assume that the claim is true for $n - 1$ (this is the induction hypothesis) and prove that it is true for n. Take any set H of n horses. We show that all the horses in this set are the same color. Remove one horse a from S to obtain the set $(H - \{a\})$ with just $n - 1$ horses. By the induction hypothesis, all the horses in $(H - \{a\})$ are the same color. Now replace the removed horse a and remove a different one b to obtain the set $(H - \{b\})$. By the same argument, all the horses in $(H - \{b\})$ are the same color. Since both a and b have the same color as all the other horses, a and b must have the same color. So all horses in H have the same color.