What Machines Can Do
Definition.

\[ A_{\text{DFA}} = \{ \langle B, w \rangle \mid B \text{ is a DFA that accepts input string } w \} \]

\[ A_{\text{NFA}} = \{ \langle B, w \rangle \mid B \text{ is an NFA that accepts input string } w \} \]

\[ A_{\text{REX}} = \{ \langle R, w \rangle \mid R \text{ is a regular expression that generates string } w \} \]
Deciding Regular Languages

Theorem. $A_{\text{DFA}}$ is decidable.

Proof. $M$ = “On input $<B, w>$,

1. Simulate $B$ on input $w$.

2. If the simulation ends in an accept state, accept. If it ends in a nonaccepting state, reject.”
Guessing is No Problem

Theorem. $A_{NFA}$ is decidable.

Proof. $N =$ “On input $< B, w>$,

1. Convert NFA $B$ to an equivalent DFA $C$.
2. Simulate TM $M$ on input $< C, w>$.
3. If $M$ accepts, accept. Otherwise, reject.”
Theorem. \( A_{\text{REX}} \) is decidable.

Proof. \( P = \) "On input \( \langle R, w \rangle \),

1. Convert RE \( R \) to an equivalent DFA \( C \).

2. Simulate TM \( M \) on input \( \langle C, w \rangle \).

3. If \( M \) accepts, accept. Otherwise, reject."

Deciding Regular Expressions
Emptiness Testing

Definition. \( E_{\text{DFA}} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset \} \)

Is \( E_{\text{DFA}} \) decidable?
Equivalence Testing

Definition. \( EQ_{DFA} = \{ \langle A, B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B) \} \)

Is \( EQ_{DFA} \) decidable?
1. For the DFA $M$ on the right
   a. Is $<M, 0100> \in A_{DFA}$?
   b. Is $<M, 010> \in A_{DFA}$?
   c. Is $<M> \in A_{DFA}$?
   d. Is $<M, 0100> \in A_{REX}$?
   e. Is $<M> \in E_{DFA}$?
   f. Is $<M, M> \in EQ_{DFA}$?

2. Let $ALL_{DFA} = \{<A> \mid A \text{ is a DFA and } L(A) = \Sigma^* \}$.
   Show that $ALL_{DFA}$ is decidable.
Deciding Context-Free Languages?

Definition. \( A_{CFG} = \{ \langle G, w \rangle \mid G \text{ is a CFG that generates } w \} \)
Theorem. \( A_{CFG} \) is decidable.

Proof. \( S = \) “On input \( <G, w> \),

1. Convert \( G \) to an equivalent grammar in Chomsky normal form.

2. List all derivations with \( 2n-1 \) steps, where \( n \) is the length of \( w \).

3. If any of these derivations generate \( w \), accept. Otherwise, reject.”
Emptiness Testing

Definition. \( E_{\text{CFG}} = \{ <G> \mid G \text{ is a CFG and } L(G) = \emptyset \} \)

Is \( E_{\text{CFG}} \) decidable?
Equivalence Testing

Definition. \( EQ_{CFG} = \{ <G, H> \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H) \} \)

Is \( EQ_{CFG} \) decidable?
The Missing Piece

Theorem. Every context-free language is decidable.

Proof. Let $G$ be a CFG for $A$. We design a TM $M_G$ that decides $A$ as follows.

$M_G = \text{"On input } w.$

1. Run TM $S$ on input $<G, w>$.
2. If this machine accepts, accept. Otherwise, reject."