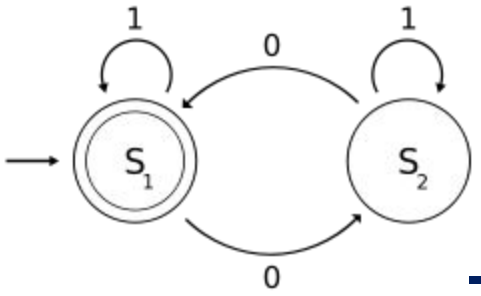
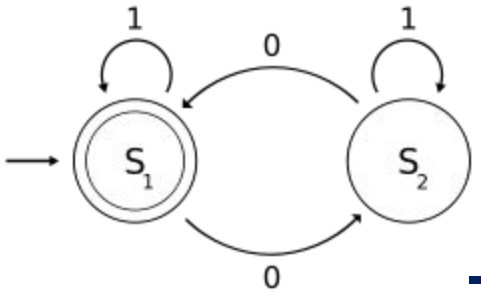



# Undecidable Problems About Languages

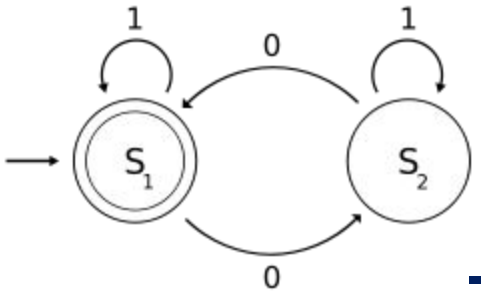
Sipser: Section 5.1 pages 215 - 226



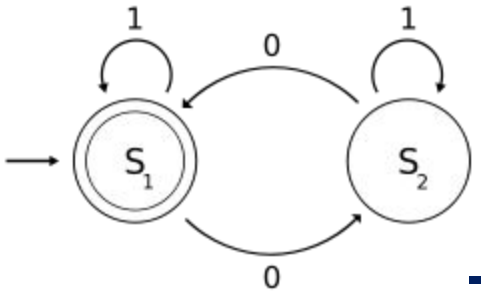

# Today's adventures

- Undecidability
- New proof technique/tool: reduction proofs!
- Prove undecidability for HALT_TM, E_TM, EQ_TM, REGULAR_TM

Weekly tip: Reflect on "be" versus "do"

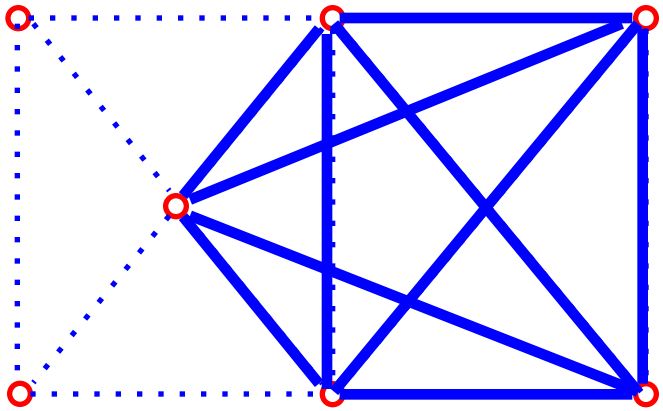



# Reducibility

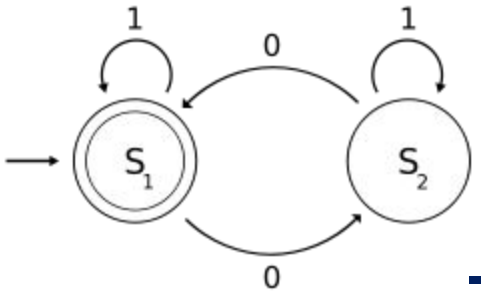



# Clique and Independent Set

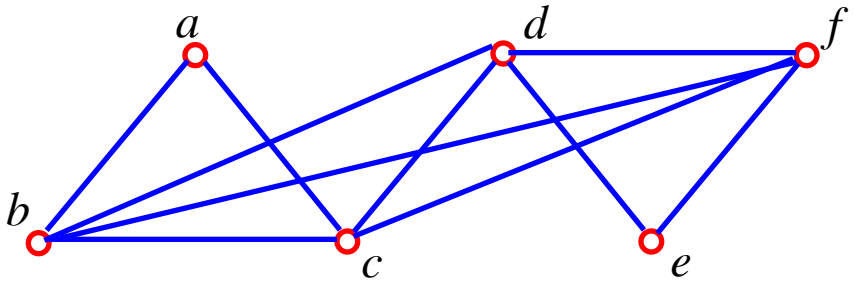
CLIQUE = {<*G*,*k*> | *G* is a graph with a *k*-clique}

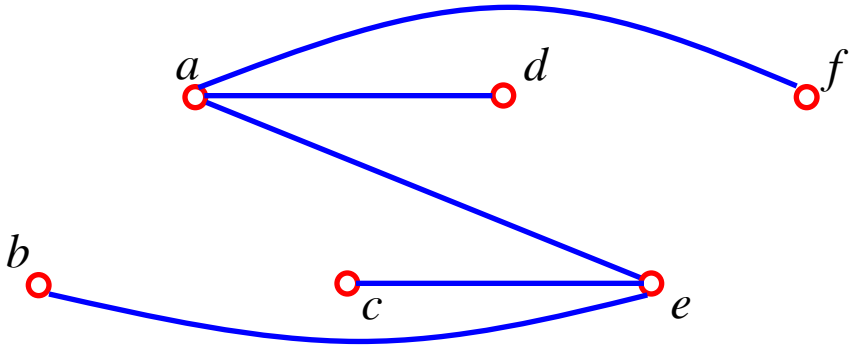


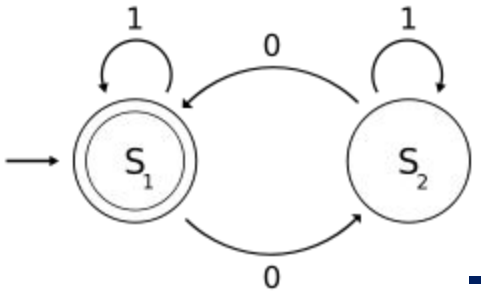
INDEPENDENT = {<*G*,*k*> | *G* is a graph containing an independent set of size *k*}




# CLIQUE reduces to INDEPENDENT



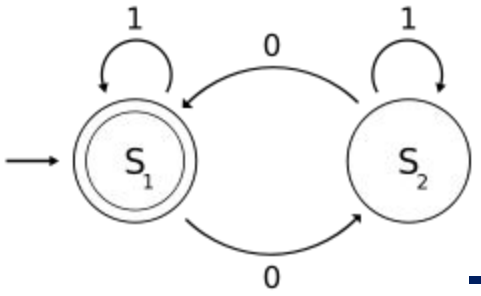






# Certified Impossible

**Theorem.** $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$ is undecidable.

**Definition.** $HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \}$

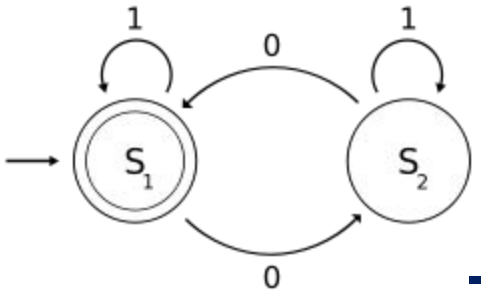



# The Halting Problem (Again!)

**Theorem.** $HALT_{TM}$ is undecidable.

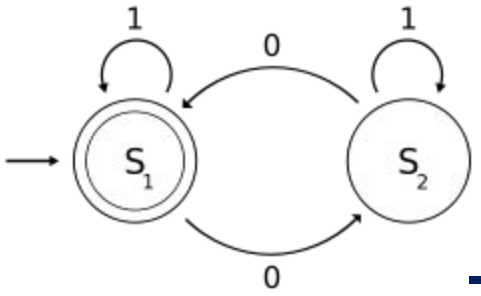
**Proof Idea.** We know $A_{TM}$ is undecidable. We need to reduce one of $HALT_{TM}$ or $A_{TM}$ to the other.

Which way to go?




# $HALT_{TM}$ is undecidable

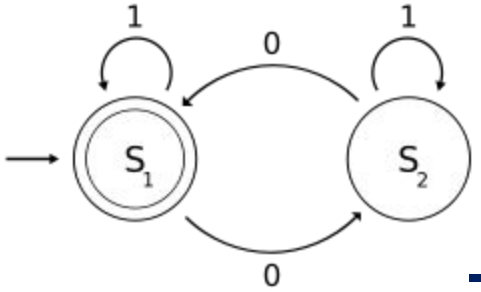
**Proof.** Suppose $R$ decides $HALT_{TM}$. Define




# Does $M$ Accept Anything at All?

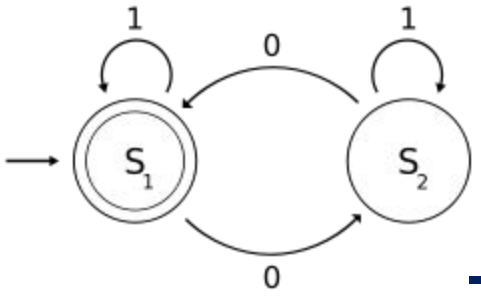
**Definition.** $E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \varnothing \}$

**Theorem.** $E_{\text{TM}}$ is undecidable.




# $E_{\text{TM}} = \{<M> \mid M \text{ is a TM \& } L(M) = \varnothing\}$

**Proof.** Given an input $<M, w>$ we construct a machine $M_w$ as follows:



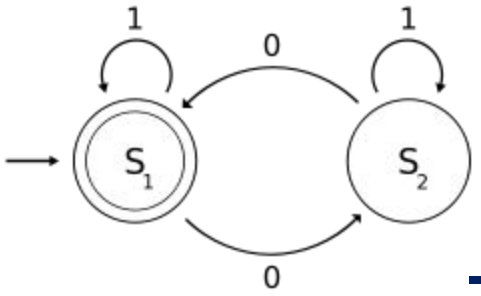

# With Power Comes Uncertainty

$M$ accepts $w$ $\quad$ $L(M) = \varnothing$ $\quad$ $L(M_1) = L(M_2)$

Turing machines

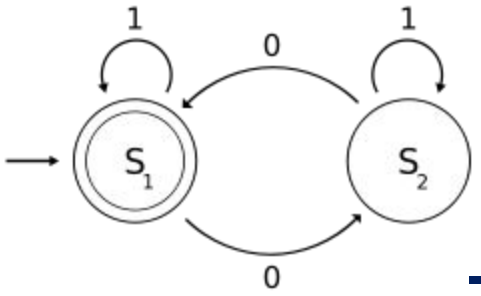
Pushdown machines

Finite machines




# It's Even Worse Than You Thought

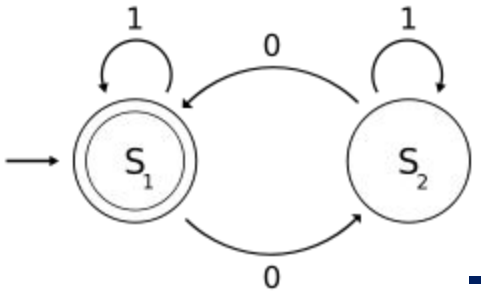
**Rice's Theorem.** Any *nontrivial property* of the languages recognized by Turing machines is undecidable.




# For Example

**Definition.** $REGULAR_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular} \}$.

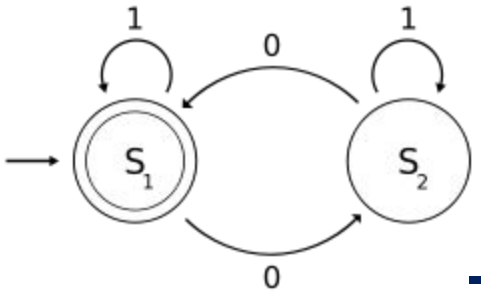
**Theorem.** $REGULAR_{TM}$ is undecidable.




# $REGULAR_{TM}$ is undecidable

**Proof.** For a contradiction, let $R$ be a TM that decides $REGULAR_{TM}$. Define

$S$ = "On input $\langle M, w \rangle$:




# Problems

- Let $EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1$ and $M_2$ are TMs and $L(M_1) = L(M_2) \}$. Show that $EQ_{TM}$ is undecidable by reducing $E_{TM}$ to $EQ_{TM}$.

- Consider the problem of determining whether a two-tape TM ever writes a nonblank symbol on its second tape when run on input *w*. Formulate this problem as a language and show that it is undecidable. (Hint: create an intermediary TM *T* that writes a nonblank symbol on its second tape iff *M* accepts *w*.)