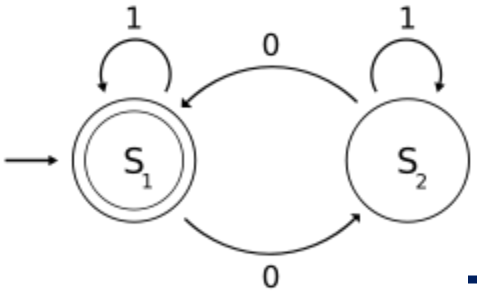
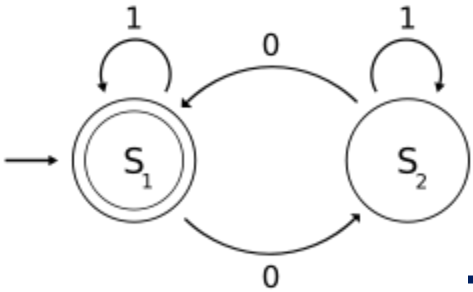


Mapping Reducibility

Sipser: Section 5.3 pages 234 - 238



$E_{TM} = \{\langle M\rangle \mid M$ is a TM & $L(M) = \emptyset\}$

Theorem. E_{TM} is undecidable.

Proof. Given an input $\langle M, w\rangle$ we construct a machine M_w that accepts a nonempty language iff M accepts w.

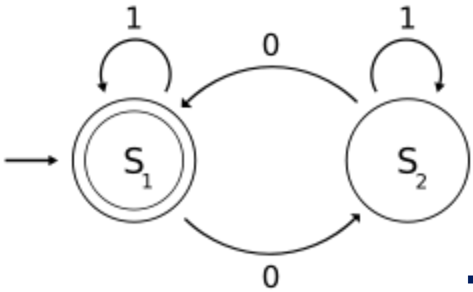
M_w = "On input x:

1. If $x \neq w$, *reject*.
2. If $x = w$, run M on input w and *accept* if M does."

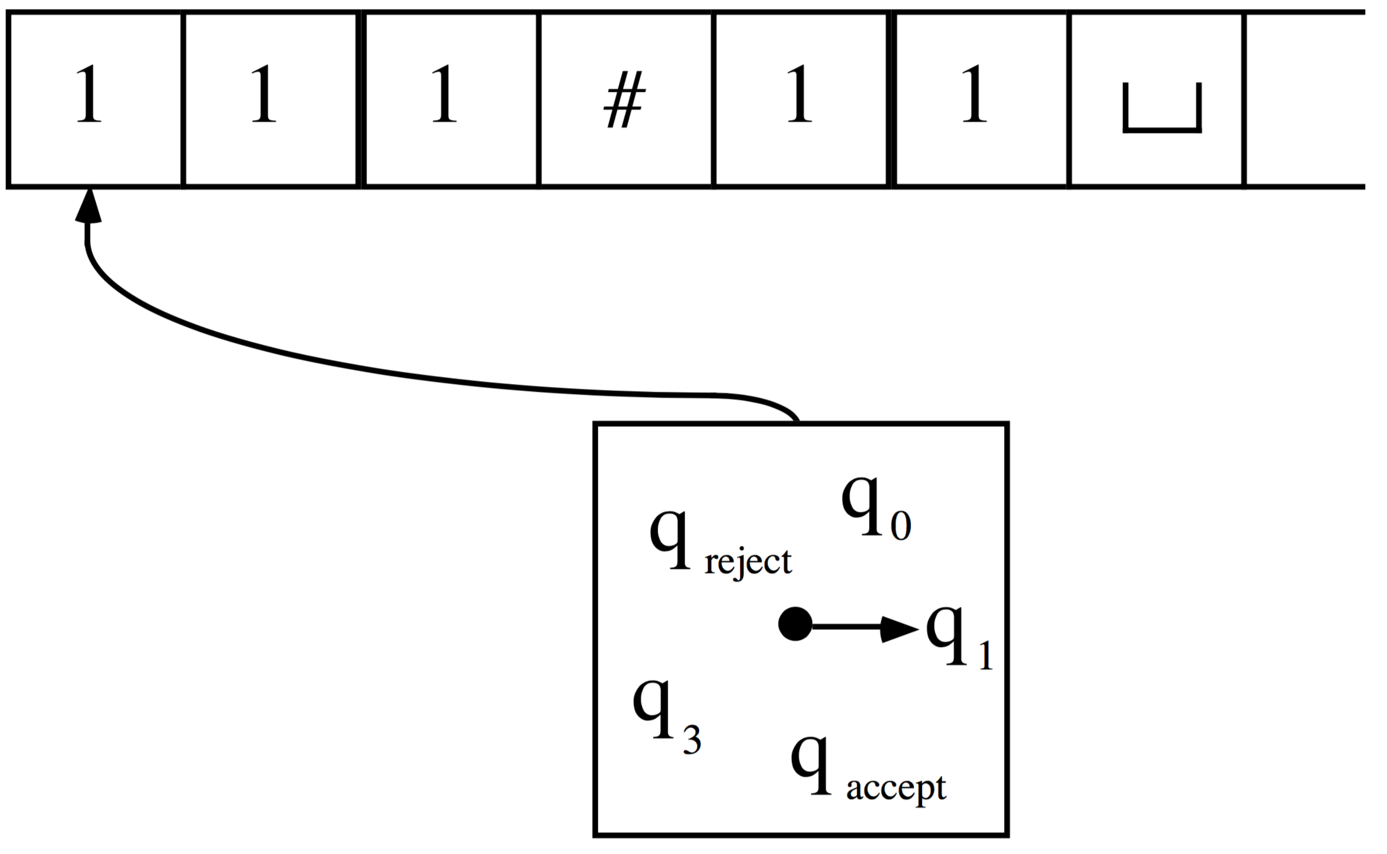
Suppose TM R decides E_{TM} and establish a contradiction by creating a decider S of A_{TM}:

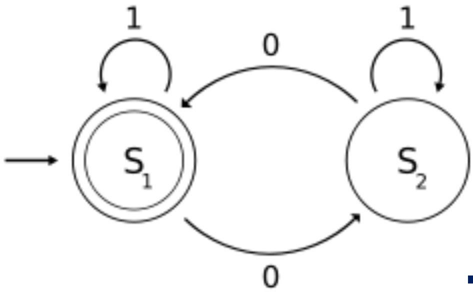
S = "On input $\langle M, w\rangle$:

1. Use the description of M and w to construct M_w.
2. Run R on input $\langle M_w\rangle$.
3. If R accepts, *reject*. If R rejects, *accept*."



Computable Functions





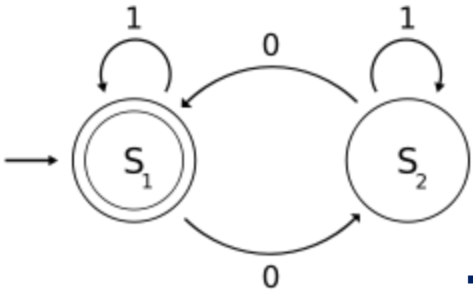
Computable Functions

Definition. A function *f*: $\Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine *M*, on every input *w*, halts with just *f*(*w*) on its tape.

Example. The *increment function*

$$inc^{++}: \{1\}^* \longrightarrow \{1\}^*$$

is Turing computable.



Machine Transformers

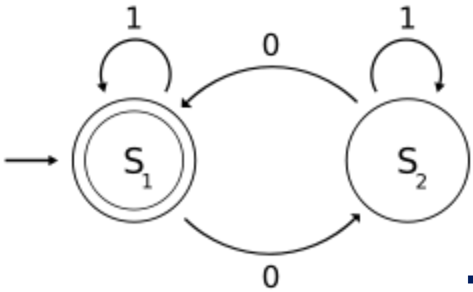
F = "On input $\langle M \rangle$:

1. Construct the machine

   M_∞ = "On input x:

   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, loop."

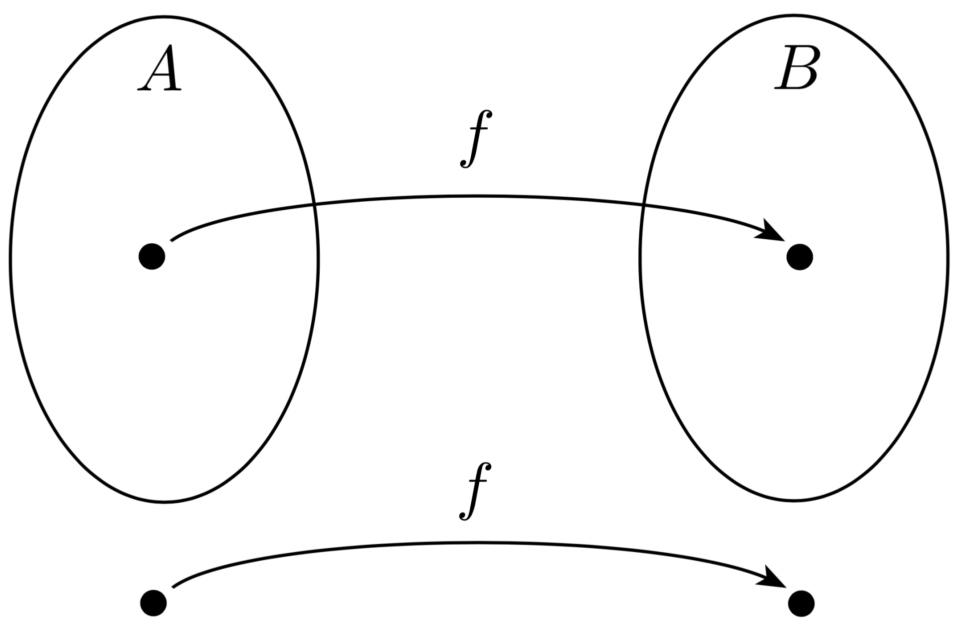
2. Output $\langle M_\infty \rangle$."

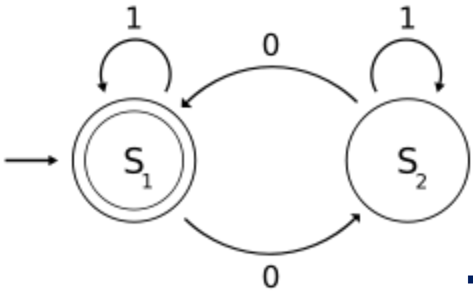


Mapping Reducibility

Definition. Language *A* is ***mapping reducible*** to language *B*, written $A \leq_m B$, if there is a computable function $f: \Sigma^* \longrightarrow \Sigma^*$, where for every *w*,

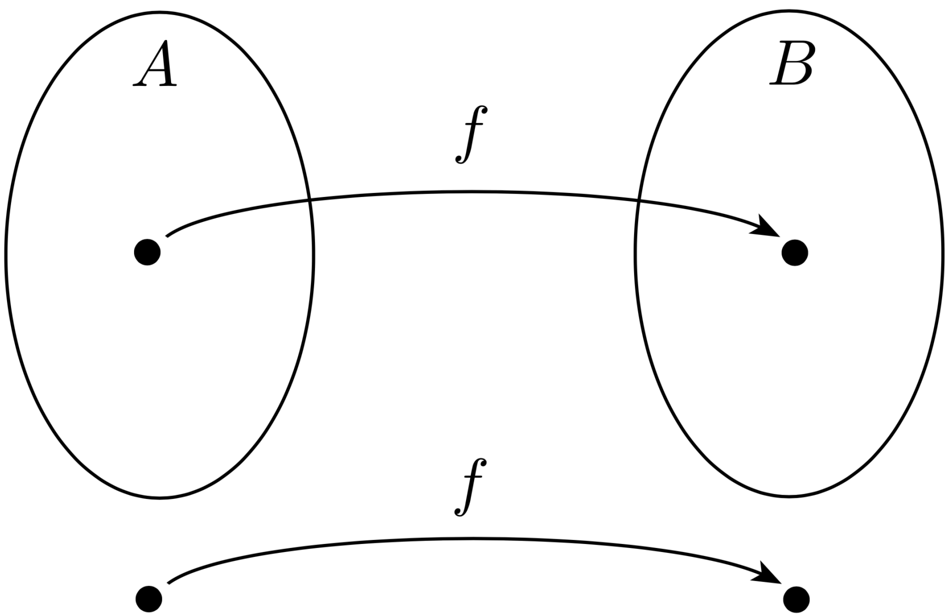
$$w \in A \Leftrightarrow f(w) \in B.$$

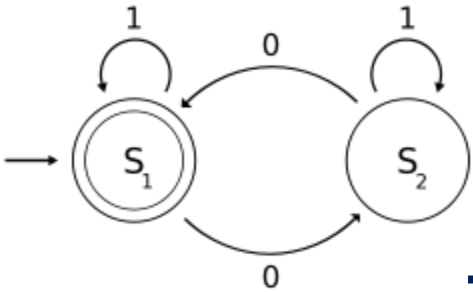




Problem Reduction

Theorem. If $A \leq_m B$ and B is decidable, then A is decidable.

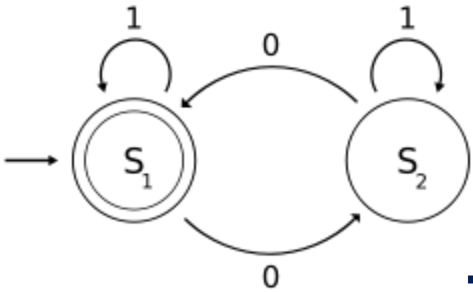




The Contrapositive is Also Useful

Theorem. If $A \leq_m B$ and B is decidable, then A is decidable.

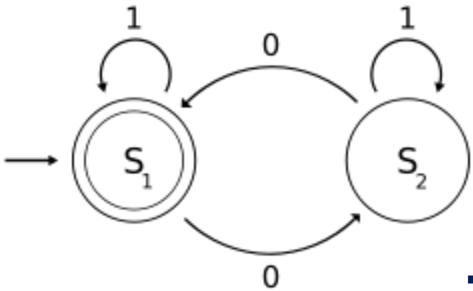
Corollary. If $A \leq_m B$ and A is undecidable, then B is undecidable.



Similarly …

Theorem. If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Corollary. If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

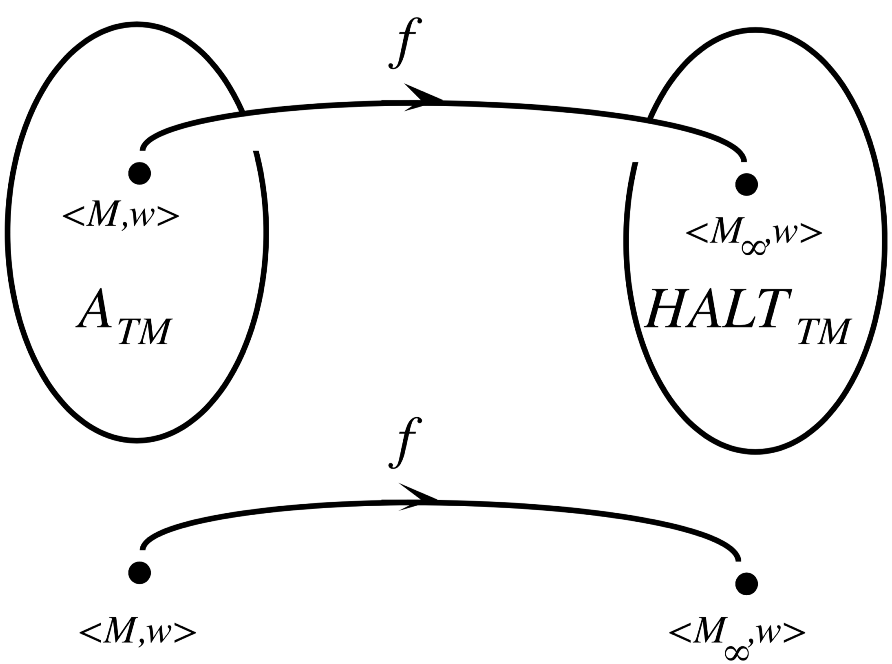


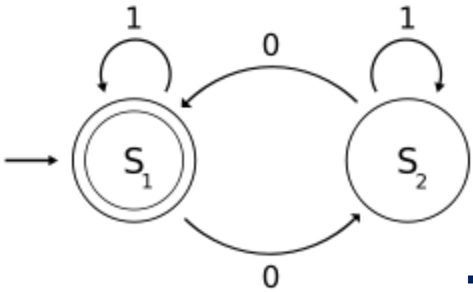
A Familiar Mapping Reduction

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

$\leq_m$

$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \}$





$A_{TM} \leq_m HALT_{TM}$

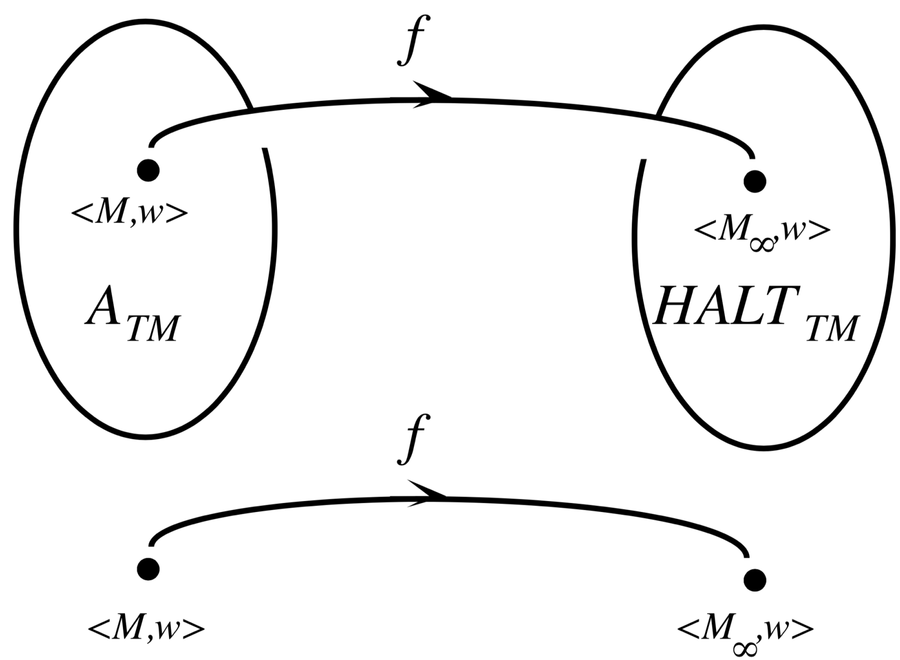
F = "On input <*M*, *w*>:

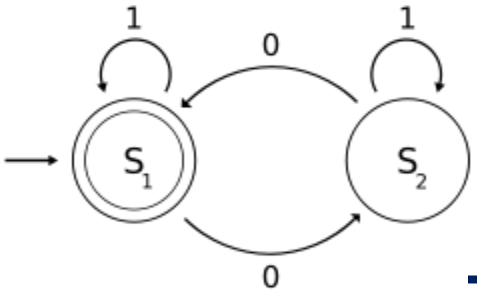
1. Construct the machine

   M_∞ = "On input *x*:

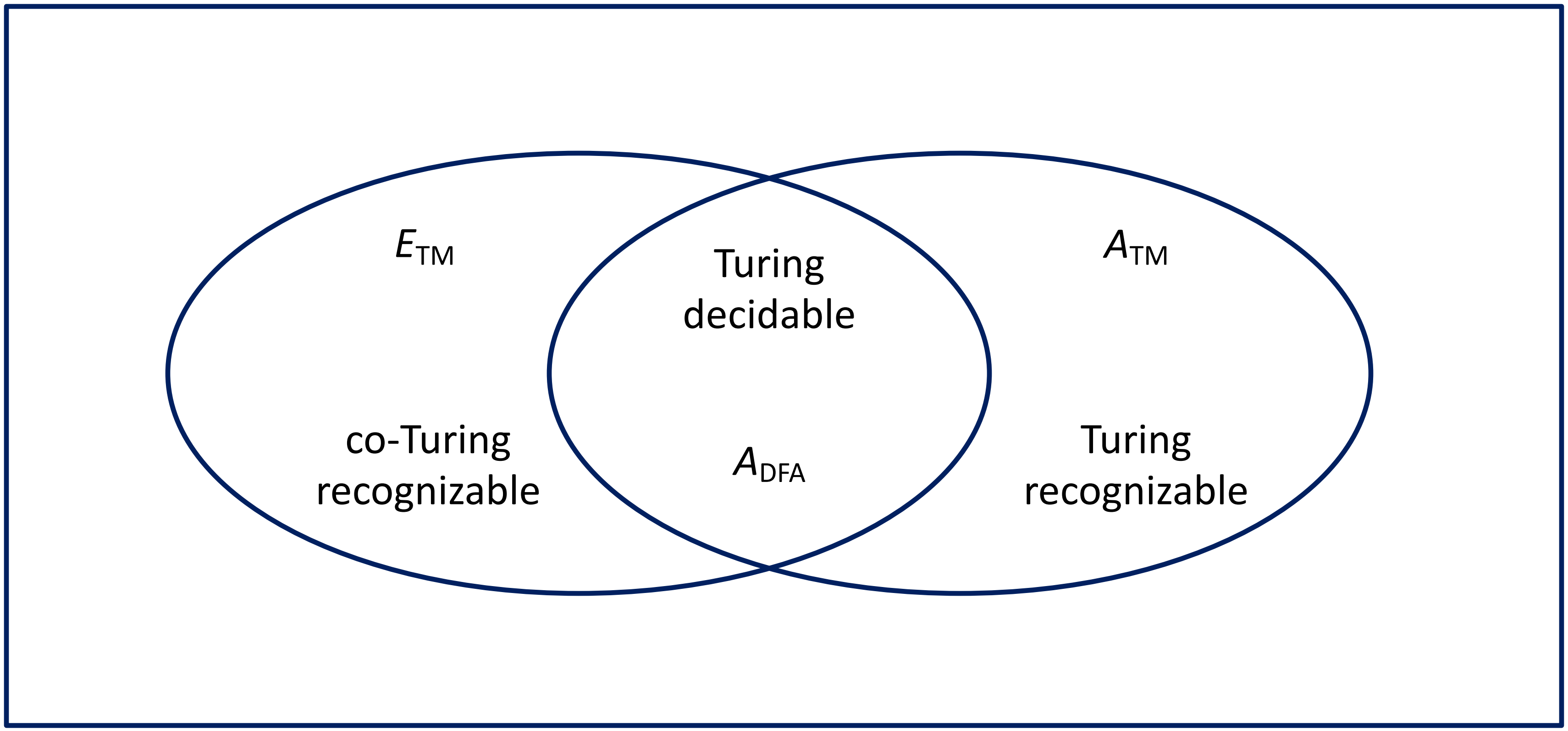
   1. Run *M* on *x*.
   2. If *M* accepts, *accept*.
   3. If *M* rejects, loop.

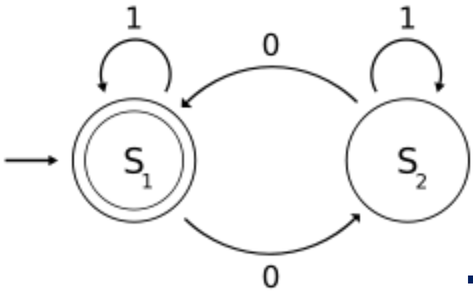
2. Output <M_∞, *w*>."





Solvable, Half-Solvable, Out-to-Lunch

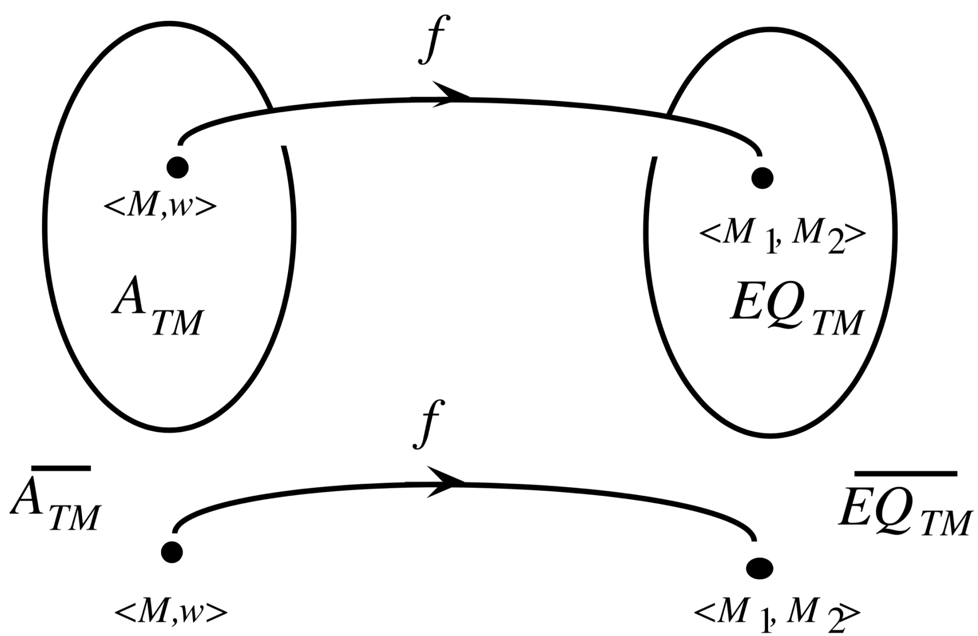


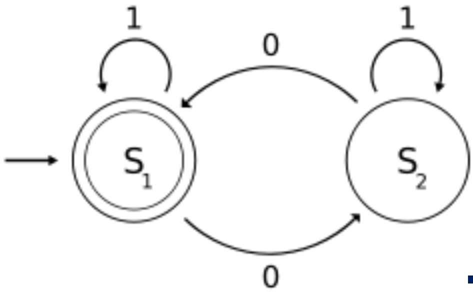


EQ_{TM} = { <M_1,M_2> | $L(M_1)=L(M_2)$ } is Out-to-Lunch

Theorem. EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

Proof. We show $A_{TM} \leq_m EQ_{TM}$. Why does this help?





$A_{TM} \leq_m EQ_{TM}$

G = "On input $\langle M, w \rangle$:

1. Construct the following two machines:

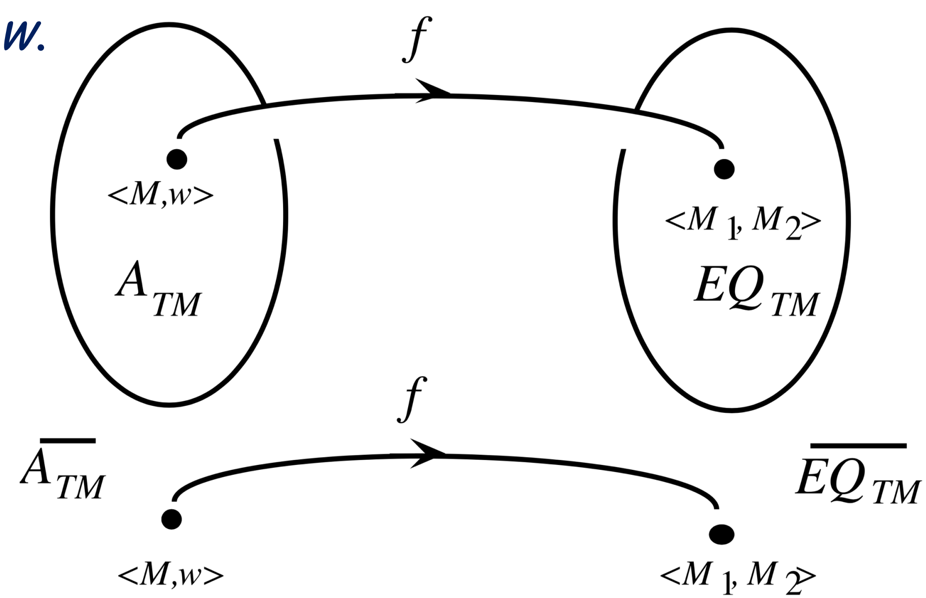
   M_1 = "On any input:

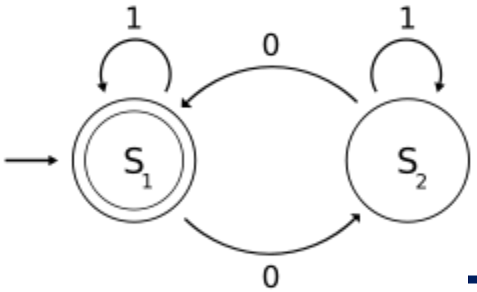
   1. *Accept*."

   M_2 = "On any input x:

   1. Ignore x and run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2 \rangle$."

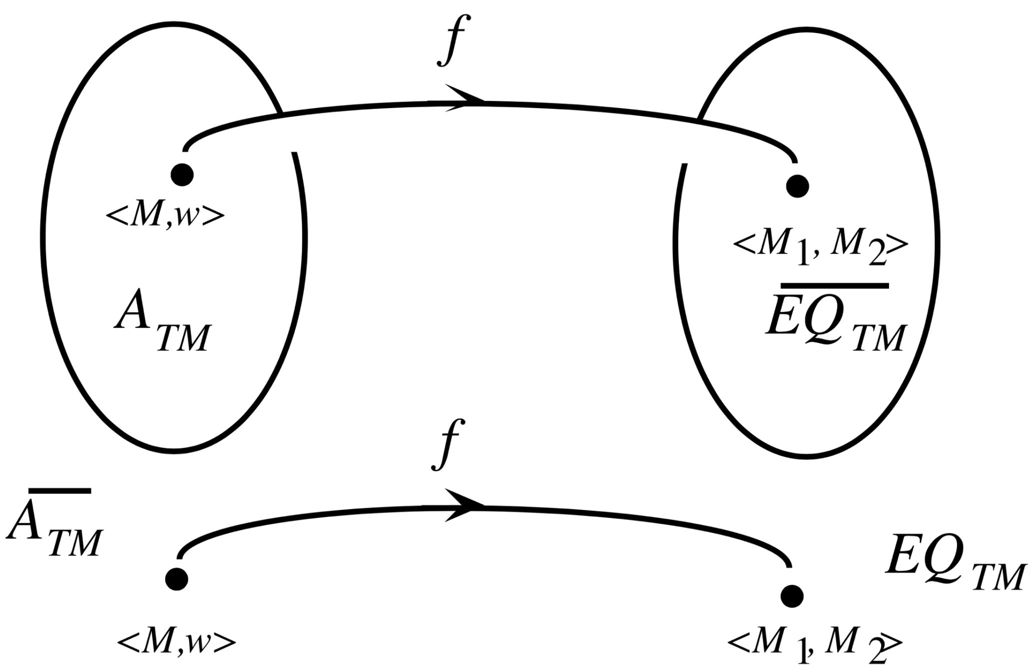


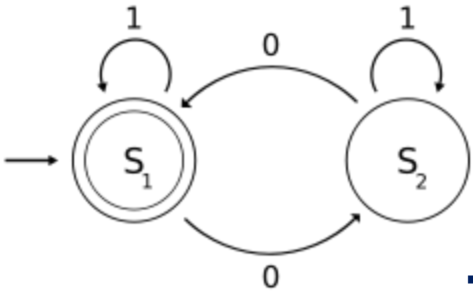


EQ_{TM} is not Turing-recognizable

Theorem. EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

Proof. We show $A_{TM} \leq_m \overline{EQ_{TM}}$.





$A_{TM} \le_m \overline{EQ_{TM}}$

F = "On input <*M*, *w*>:

1. Construct the following two machines:

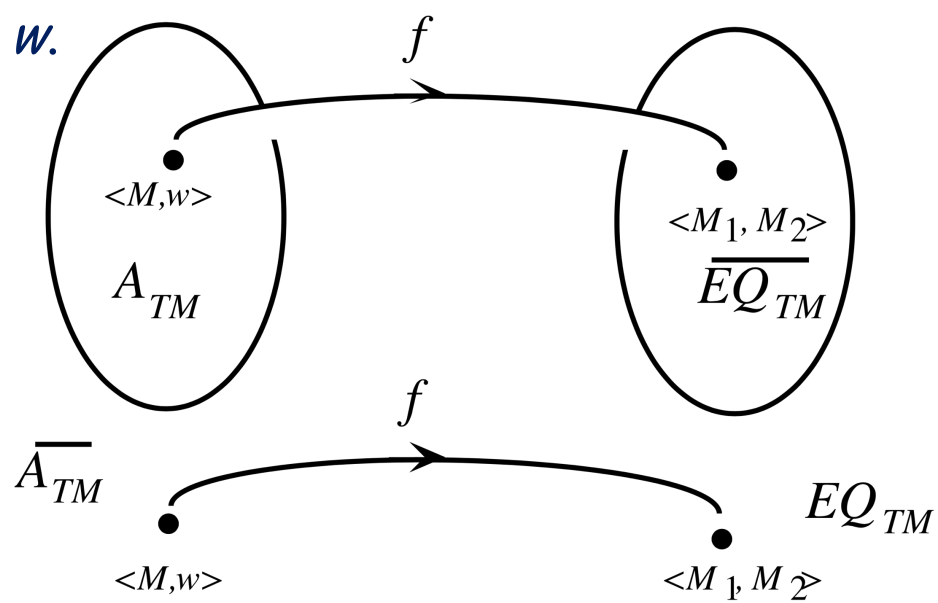
   M_1 = "On any input:

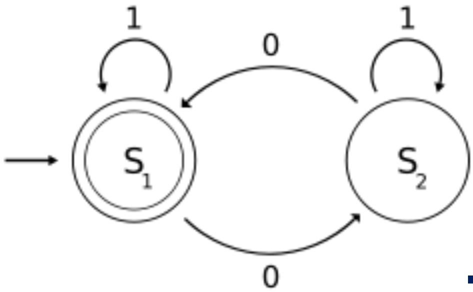
   1. *Reject*."

   M_2 = "On any input *x*:

   1. Ignore *x* and run *M* on *w*. If it accepts, *accept*."

2. Output <M_1, M_2>."





Exercises

1. Show that A_{TM} is not mapping reducible to E_{TM}.
   (Hint: Use the fact that $\overline{A_{TM}}$ is not Turing-recognizable whereas $\overline{E_{TM}}$ is Turing-recognizable.)

2. Show that if P is Turing-recognizable and $P \leq_m \overline{P}$, then P is decidable.