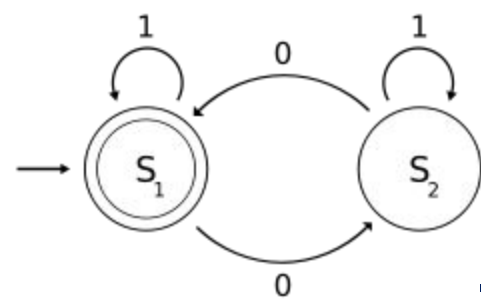
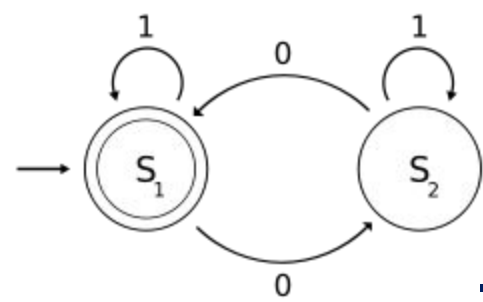




The Hardest Problem In The World

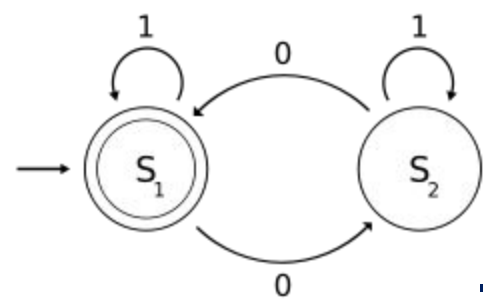


Today's adventures



- Polynomial time reductions
- CNF reduces to Clique
- NP's Hardest Problems
- Cook's Theorem
- NP-Complete Problems

Weekly tip: Be a detective for joy and fun



Warmup

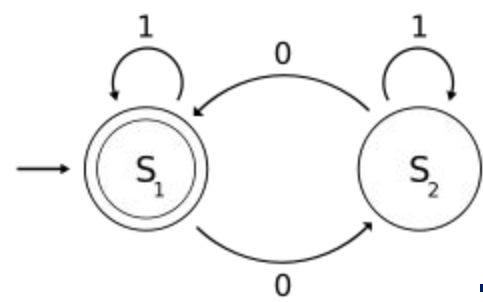
Which of the following formulas is in conjunctive normal form?

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) \wedge (x_1 \vee x_7)$$

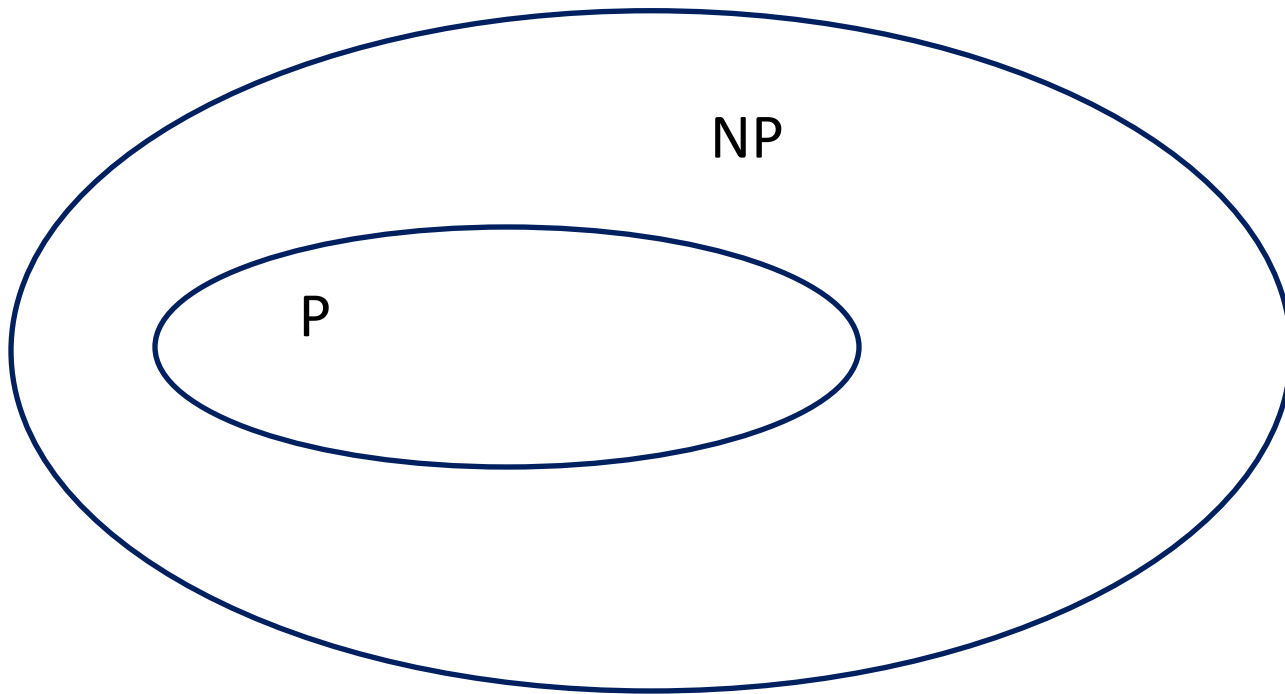
$$(x_1 \vee x_2) \wedge (x_1 \wedge x_2) \wedge (x_1 \vee x_5)$$

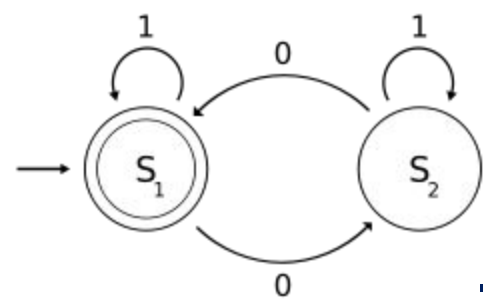
$$(\overline{x_1} \vee \overline{x_2}) \vee (x_1 \vee x_2) \wedge (x_7 \vee x_2)$$

$$(x_1 \vee x_2) \wedge (x_1 \vee x_2) \wedge (x_3 \vee x_4 \vee x_5)$$



The Classes P and NP?

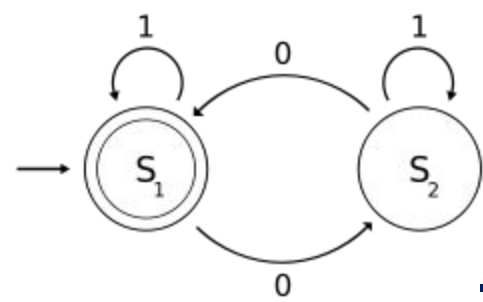




A Famous NP Problem

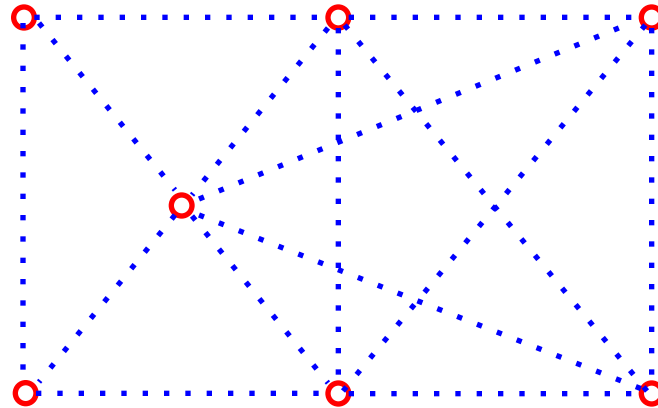
CNF satisfiability (CNFSat): given a Boolean formula B in conjunctive normal form (CNF), is there a truth assignment that satisfies B ?

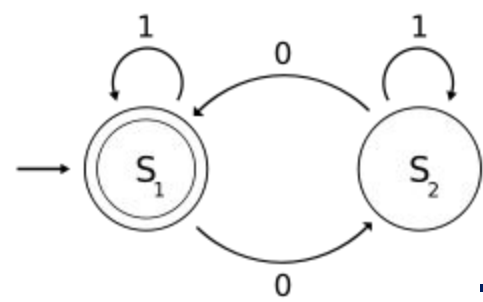
$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee \overline{x_2})$$



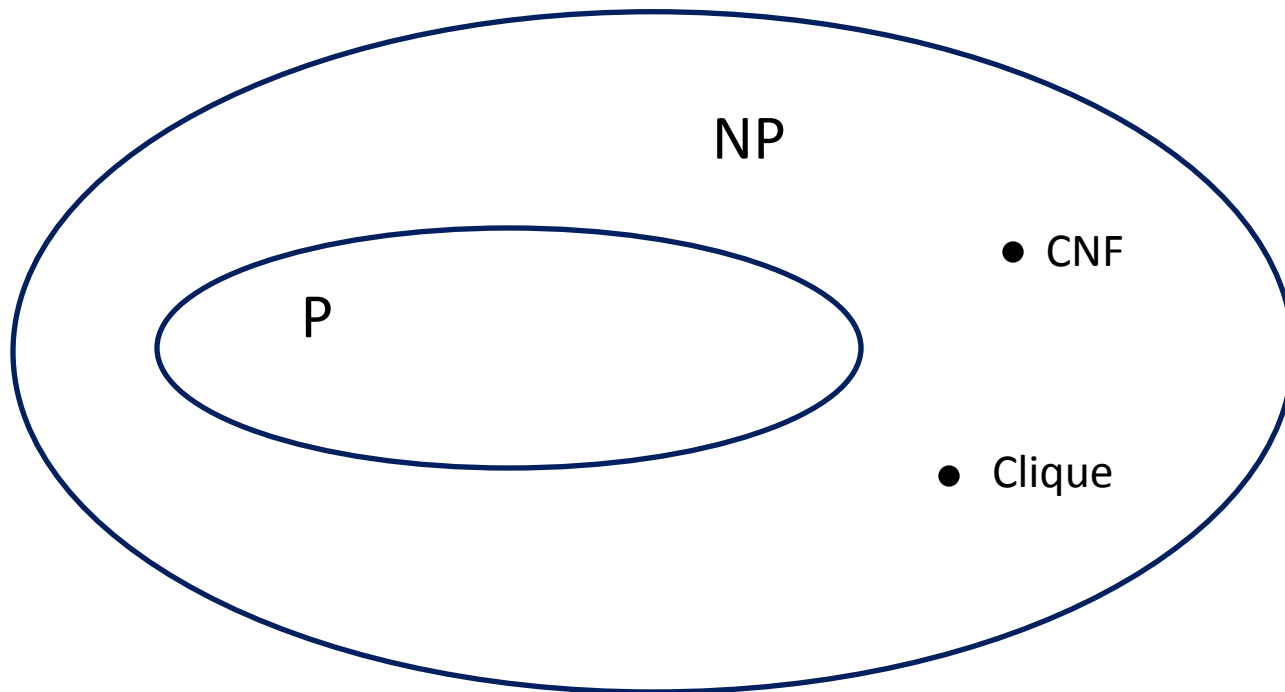
A Graph Theory NP Problem

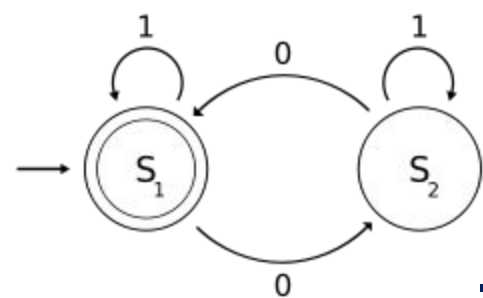
CLIQUE: given a graph $G = (V, E)$ and an integer k , does G contain C_k as a subgraph?





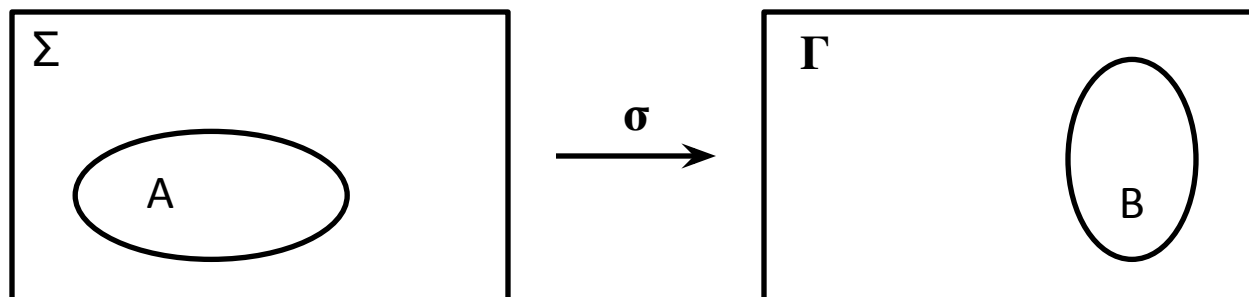
Which Problem is Harder?





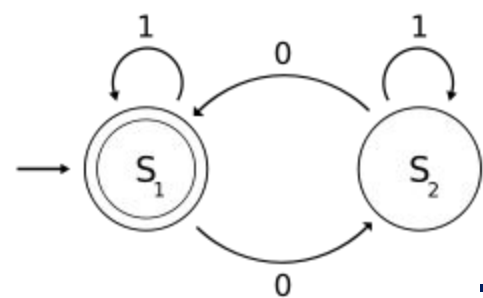
Polynomial Time Reduction

Definition. Let $A \subseteq \Sigma$ and $B \subseteq \Gamma$ be decision problems.



We write $A \leq_p B$ and say that A *reduces to* B in *polynomial time* if there is a polynomial time computable function $\sigma: \Sigma \rightarrow \Gamma$ such that for all problem instances $x \in \Sigma$,

$$x \in A \text{ iff } \sigma(x) \in B.$$

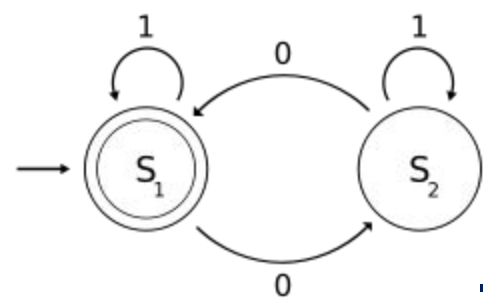


$\text{CNF} \leq_p \text{Clique}$

- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

$$\begin{array}{ccc}
 c_1 & c_2 & c_3 \\
 (x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \overline{\vee} x_2)
 \end{array}$$

the construction would yield

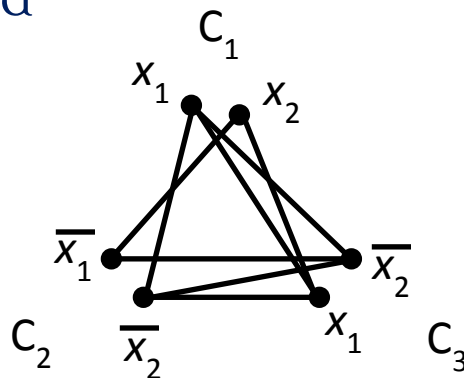


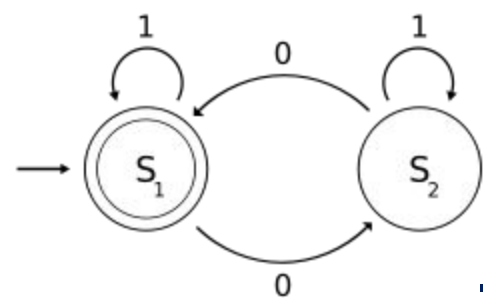
$\text{CNF} \leq_p \text{Clique}$

- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

$$\begin{array}{ccc}
 C_1 & C_2 & C_3 \\
 (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge (x_1 \bar{\vee} x_2)
 \end{array}$$

the construction would yield



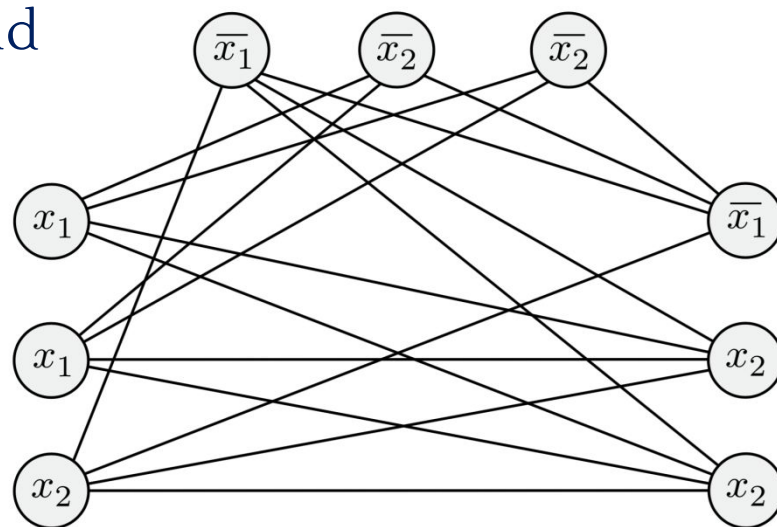


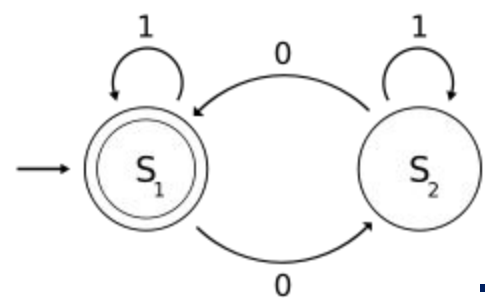
$\text{CNF} \leq_p \text{Clique}$

- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

$$(x_1 \vee \bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$$

the construction would yield

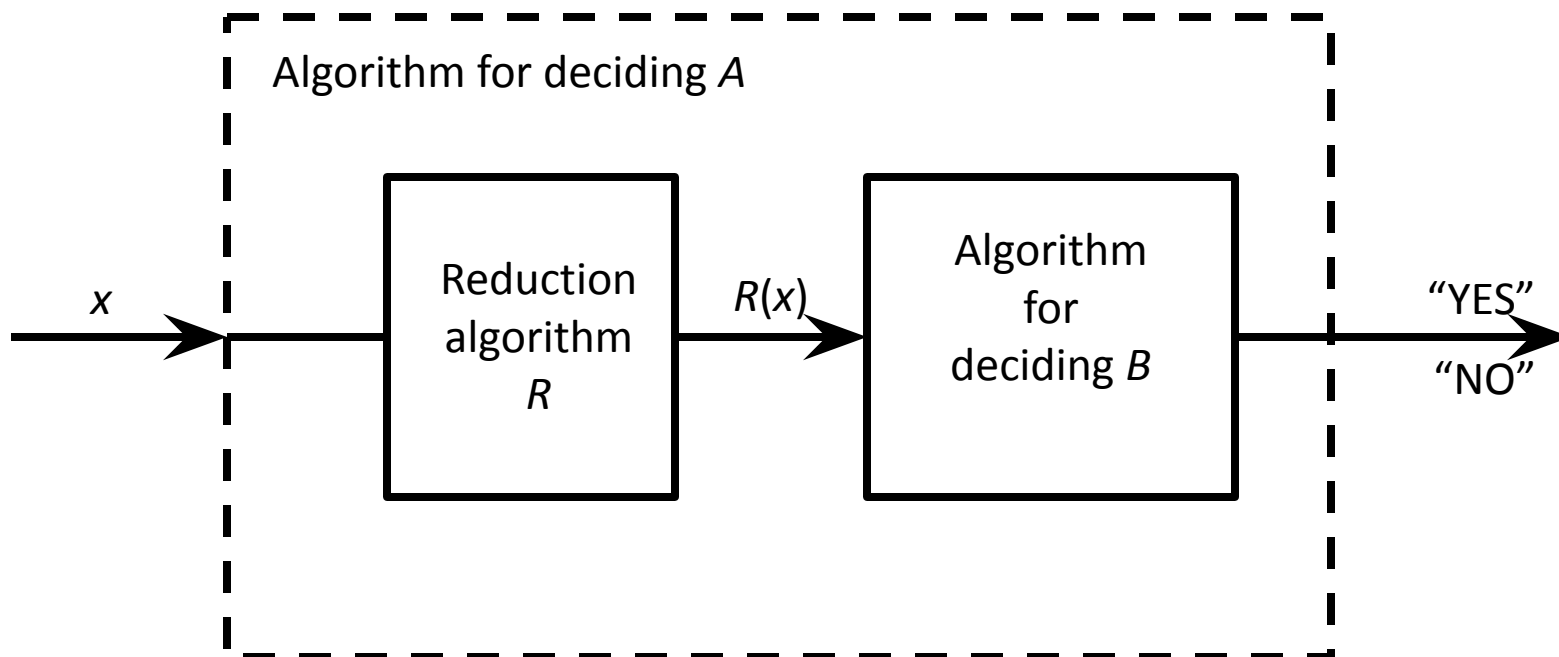


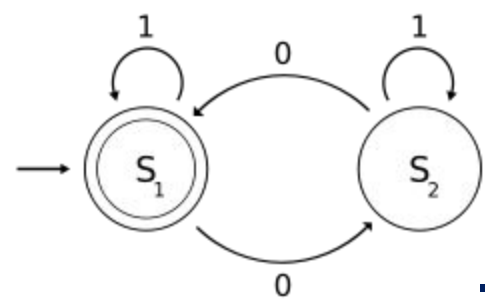


Intuitively, A is No Harder than B

Theorem. If $A \leq_p B$ and B has a polynomial time algorithm, then so does A .

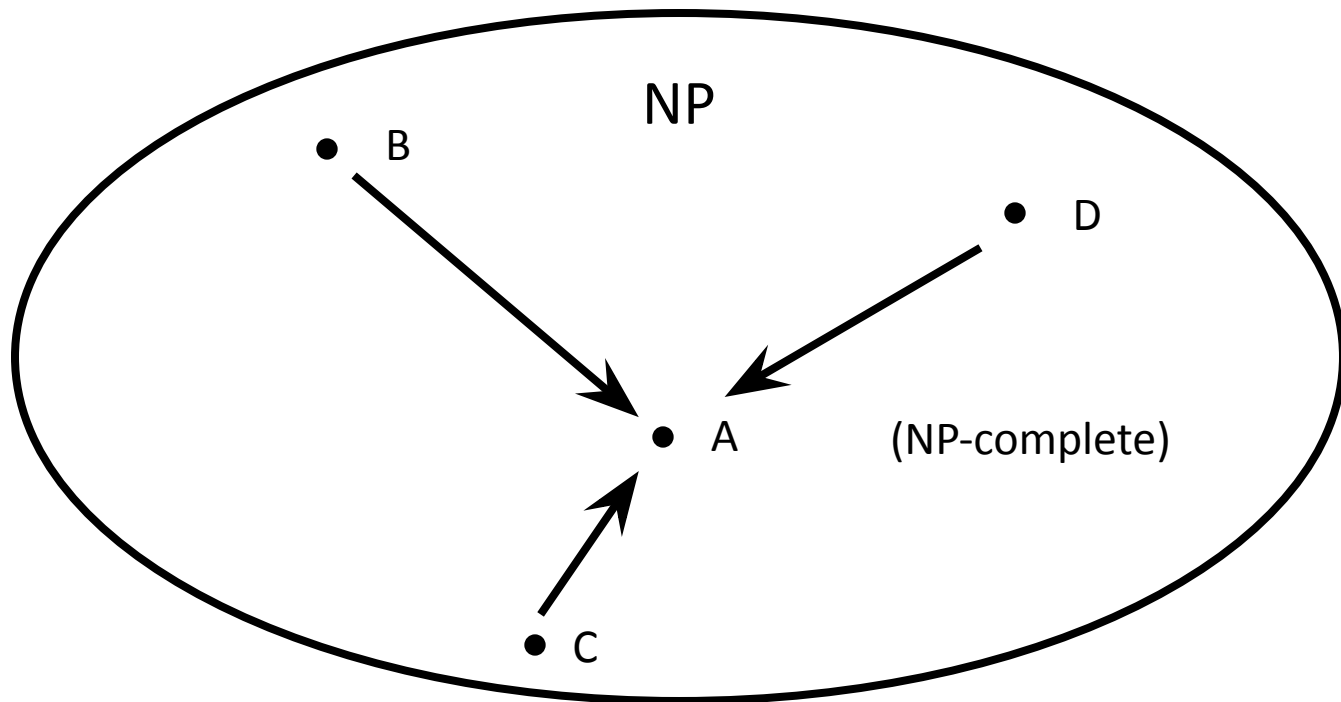
Proof.

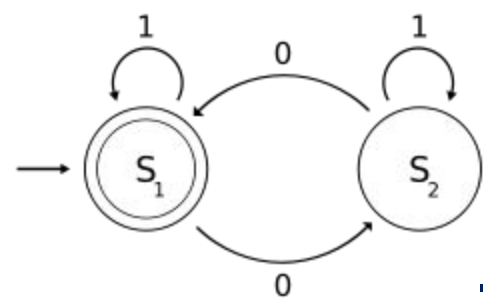




NP's Hardest Problems

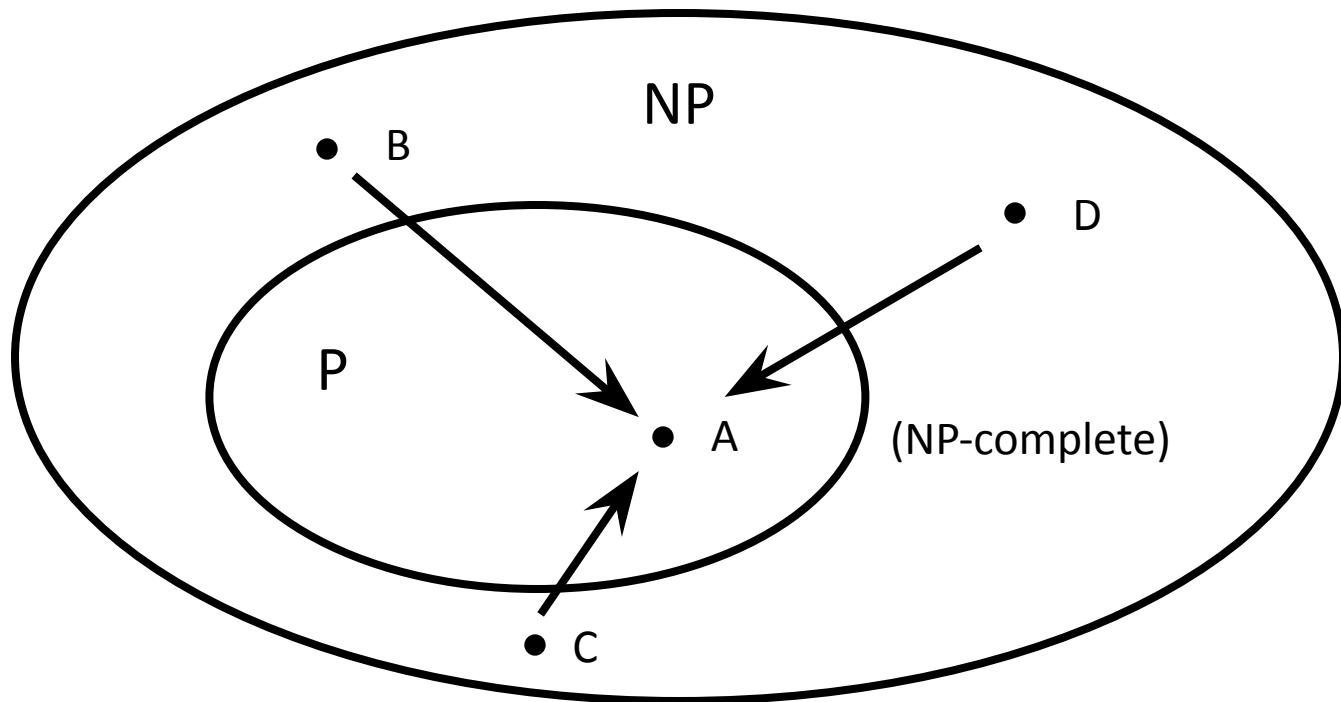
Definition. The set $A \in \text{NP}$ is *NP-complete* if for all $B \in \text{NP}$, $B \leq_p A$.

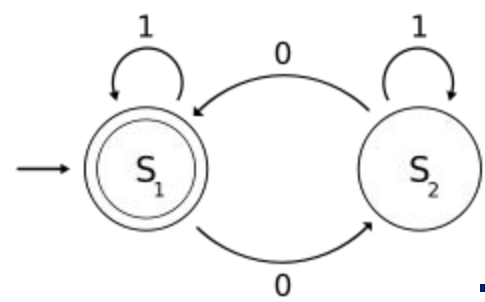




$P = NP?$

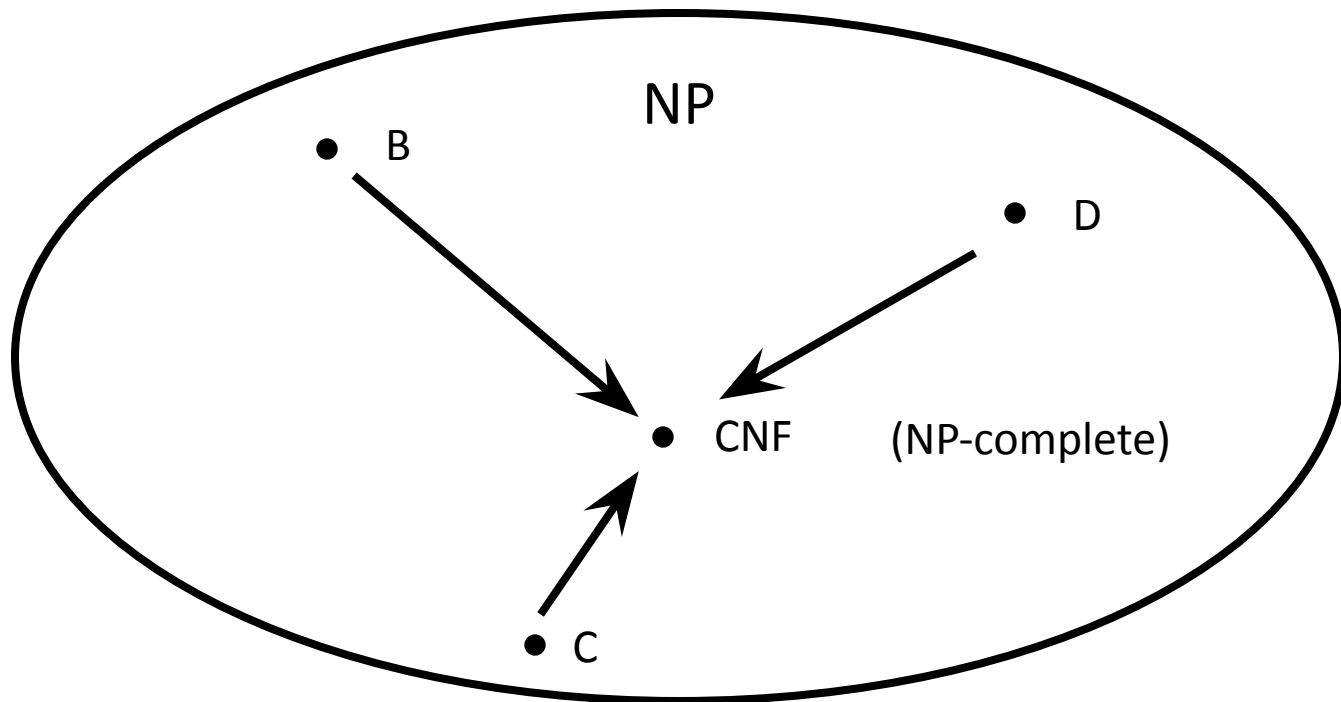
Theorem. If A is NP-complete, then $A \in P$ if and only if $P = NP$.

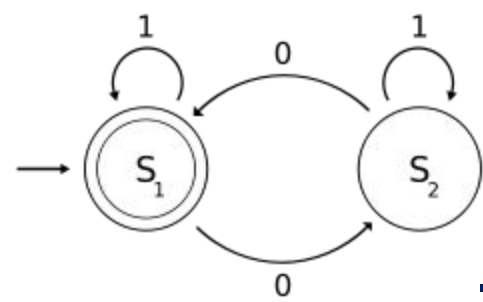




Cook's Theorem

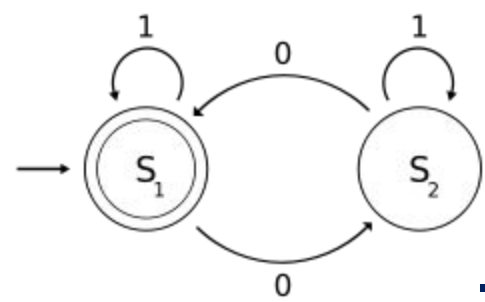
Theorem. If $A \in \text{NP}$ then $A \leq_p \text{CNFSat}$.



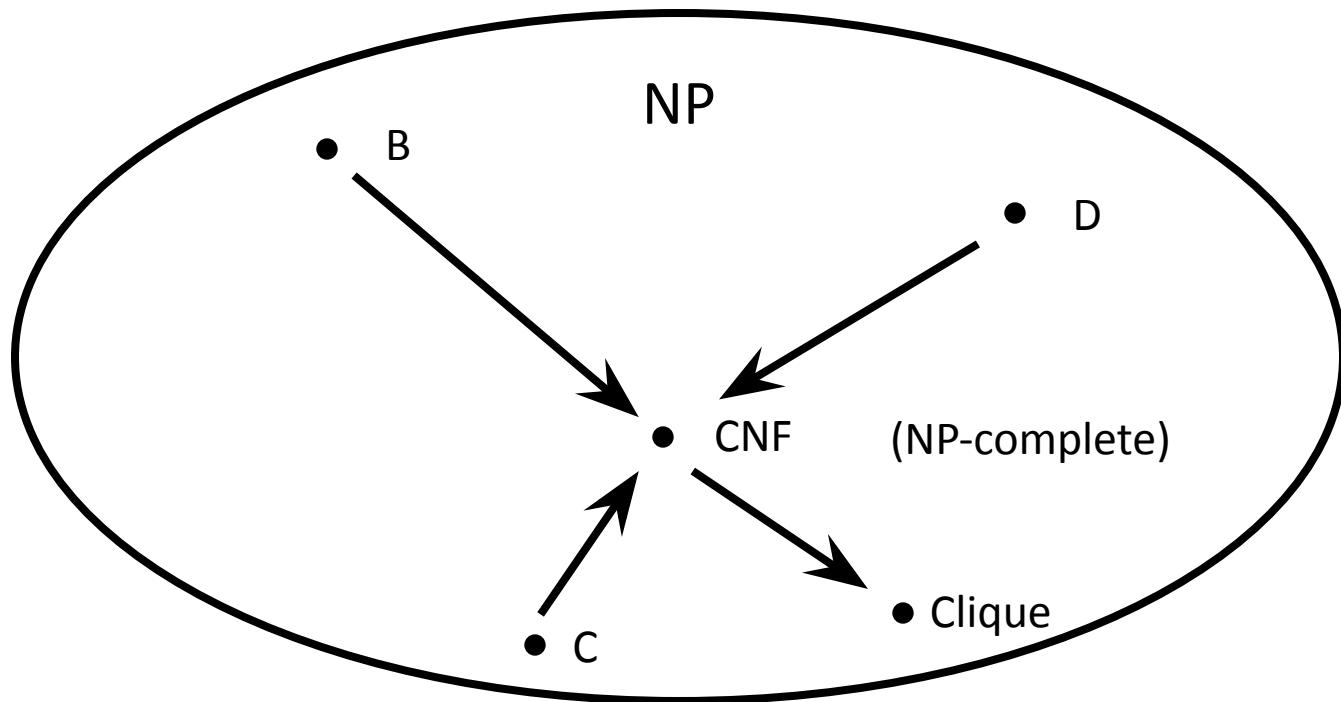


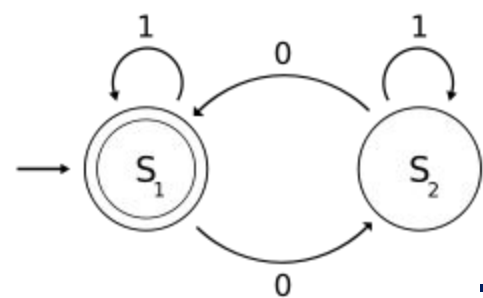
So What?

- The existence of one “natural” NP-complete problem is an interesting fact for the computer scientist.
- The existence of thousands of “natural” NP-complete problems is an essential fact for the computer scientist.



Clique is NP-Complete





NP-Complete Problems

$DOUBLE-SAT = \{ \langle \varphi \rangle \mid \varphi \text{ has at least two satisfying assignments} \}$

Show $DOUBLE-SAT$ is in NP.

Show $DOUBLE-SAT$ is NP-complete.

HINT: Reduce SAT to $DOUBLE-SAT$. Create a new Boolean formula φ' based on Boolean formula φ such that φ is in SAT iff φ' is in $DOUBLE-SAT$.

$HALF-CLIQUE = \{ \langle G \rangle \mid G \text{ has a clique of size } m/2 \text{ where } m \text{ is the number of nodes in } G \}$

Show $HALF-CLIQUE$ is in NP.

Show $HALF-CLIQUE$ is NP-complete.

HINT: Reduce $CLIQUE$ to $HALF-CLIQUE$. Create a graph H such that $\langle G, k \rangle$ is in $CLIQUE$ iff $\langle H \rangle$ is in $HALF-CLIQUE$. Consider the three cases when $k = m/2$, $k > m/2$, and $k < m/2$.