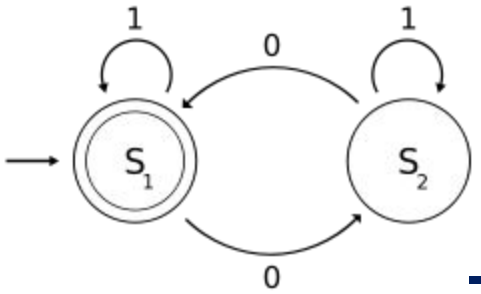


The Hardest Problem In The World

Sipser: Section 7.4 pages 299 – 311

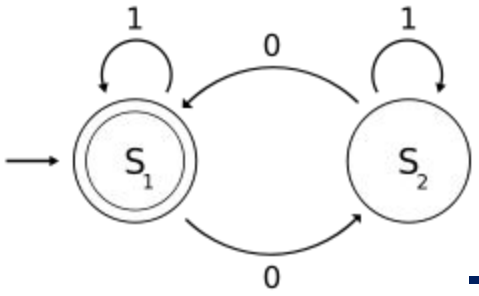


Today's adventures



- Polynomial time reductions
- CNF reduces to Clique
- NP's Hardest Problems
- Cook's Theorem
- NP-Complete Problems

Weekly tip: Be a detective for joy and fun



Warmup

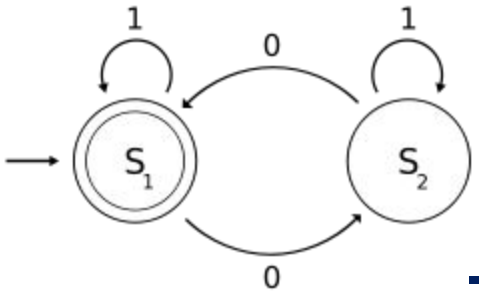
Which of the following formulas is in conjunctive normal form?

$(x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2) \wedge (x_1 \vee x_1)$

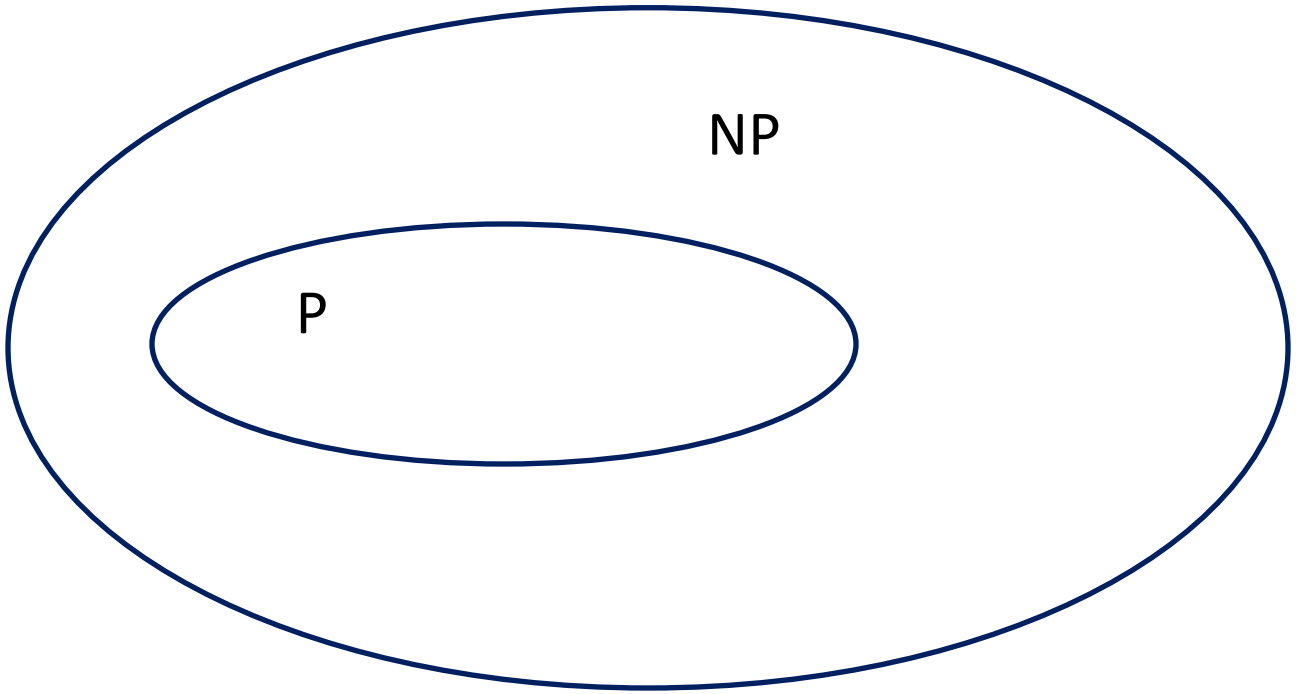
$(x_1 \vee x_2) \wedge (x_1 \wedge x_2) \wedge (x_1 \vee x_5)$

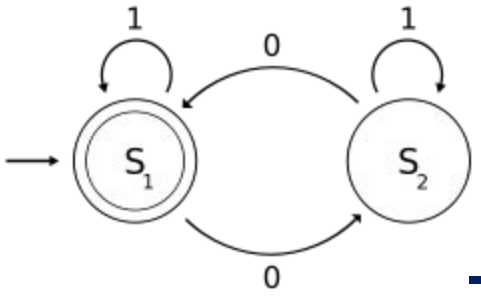
$(\overline{x}_1 \vee \overline{x}_2) \vee (x_1 \vee x_2) \wedge (x_1 \vee x_2)$

$(x_1 \vee x_2) \wedge (x_1 \vee x_2) \wedge (x_3 \vee x_4 \vee x_5)$



The Classes P and NP?

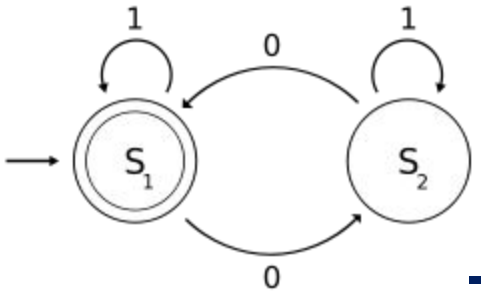




A Famous NP Problem

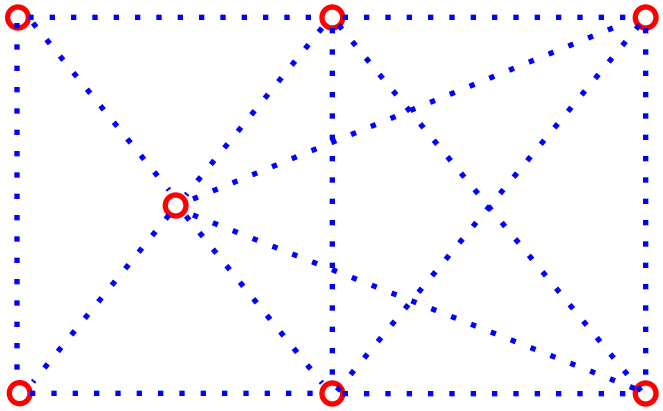
CNF satisfiability (*CNFSat*): given a Boolean formula B in conjunctive normal form (CNF), is there a truth assignment that satisfies B?

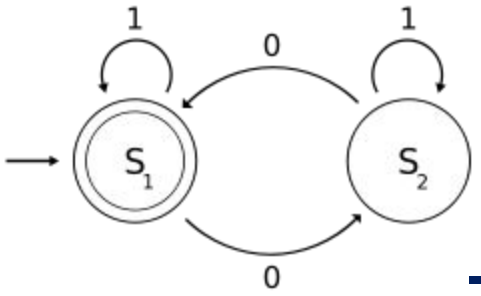
$$(x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2) \wedge (x_1 \vee \overline{x}_2)$$



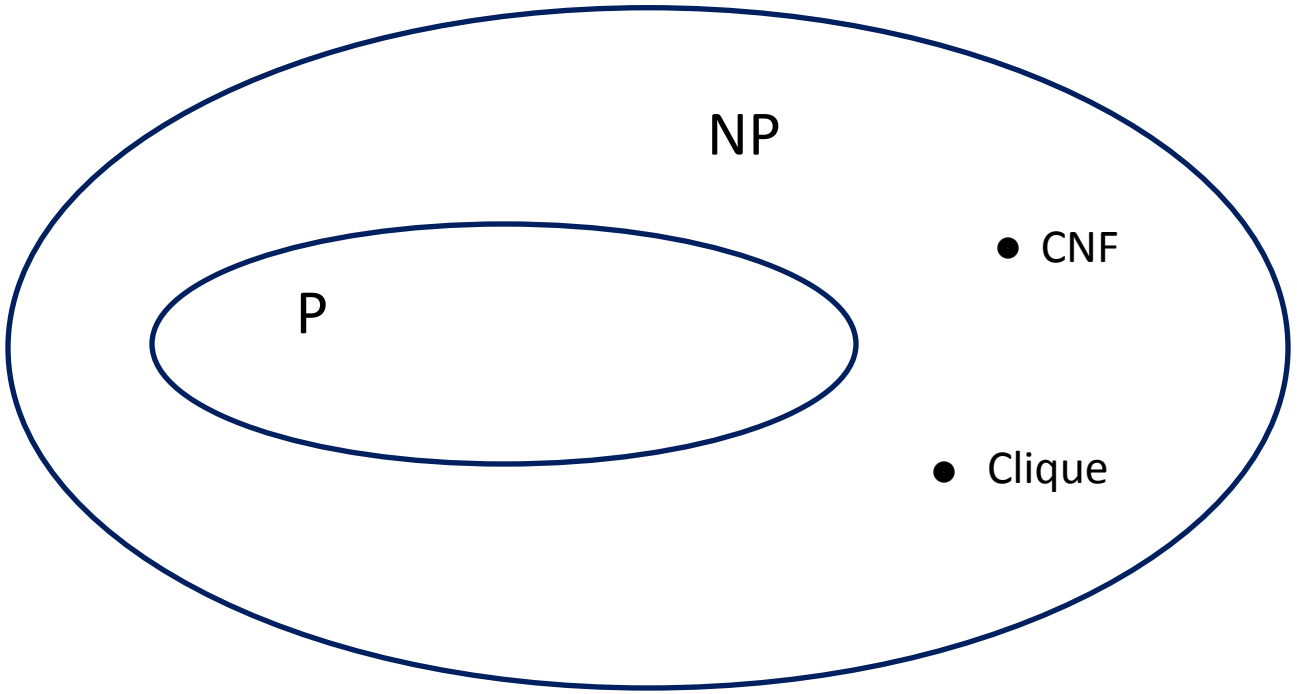
A Graph Theory NP Problem

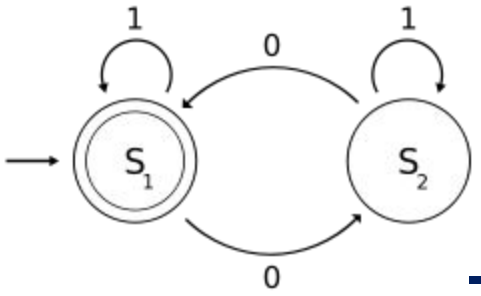
CLIQUE: given a graph $G = (V, E)$ and an integer k, does G contain C_k as a subgraph?





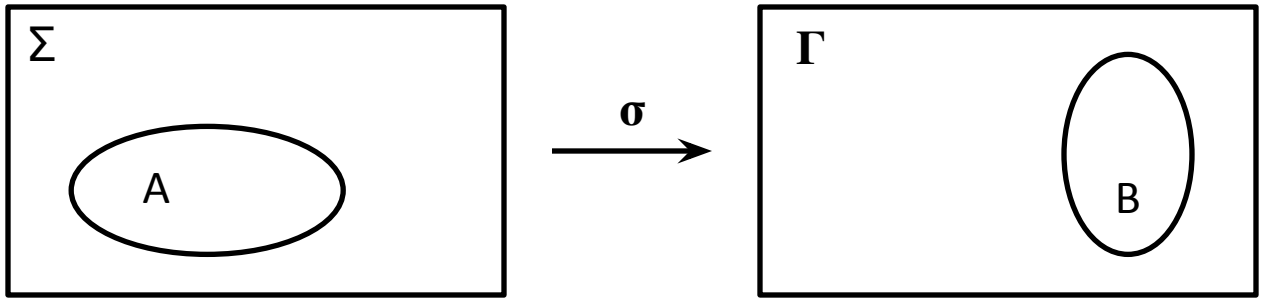
Which Problem is Harder?





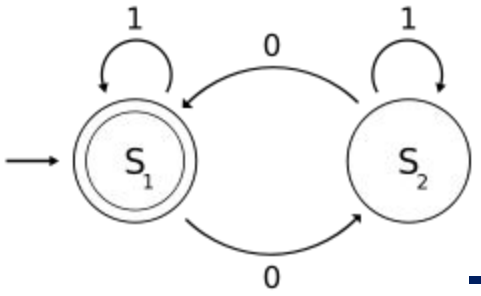
Polynomial Time Reduction

Definition. Let $A \subseteq \Sigma$ and $B \subseteq \Gamma$ be decision problems.



We write $A \leq_p B$ and say that *A reduces to B in polynomial time* if there is a polynomial time computable function $\sigma: \Sigma \rightarrow \Gamma$ such that for all problem instances $x \in \Sigma$,

$$x \in A \text{ iff } \sigma(x) \in B.$$

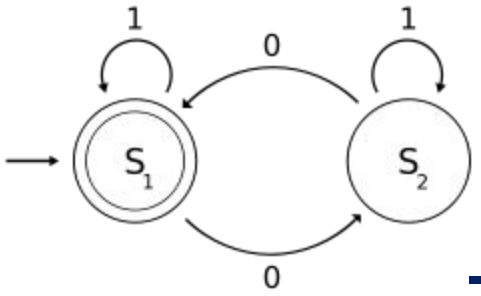


CNF $\leq_p$ Clique

- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

$$\underset{C_1}{(x_1 \vee x_2)} \wedge \underset{C_2}{(\overline{x_1} \vee \overline{x_2})} \wedge \underset{C_3}{(x_1 \overline{\vee} x_2)}$$

the construction would yield



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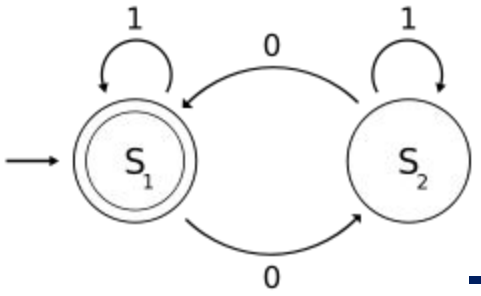
- Given

$$\underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2})}_{C_2} \wedge \underbrace{(x_1 \overline{\vee} x_2)}_{C_3}$$

the construction would yield





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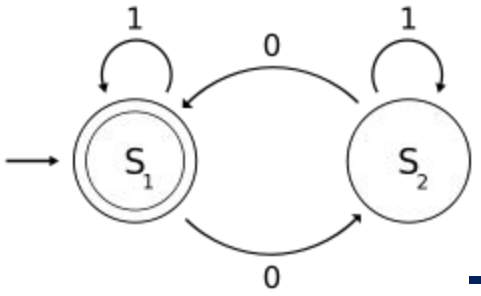
- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.
- Given

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (x_1 \vee x_2 \vee x_2)$$

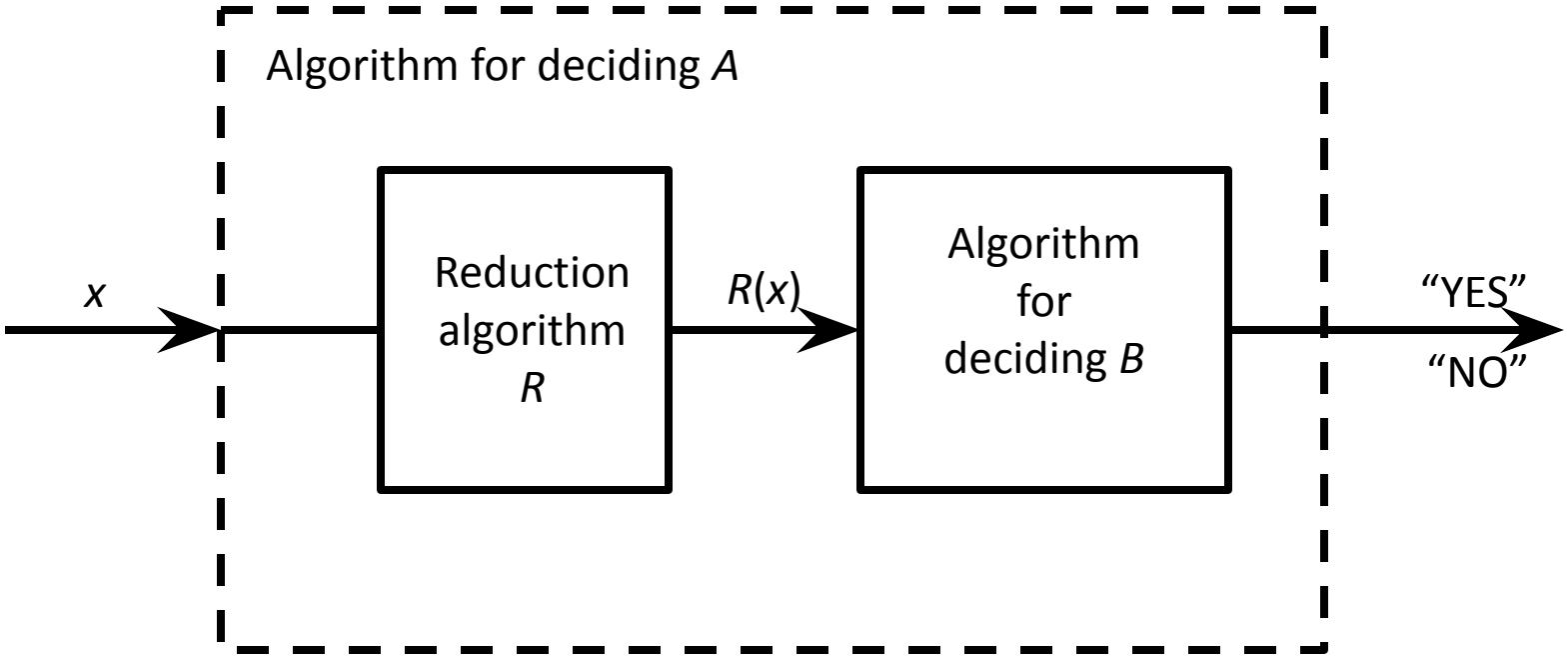
the construction would yield

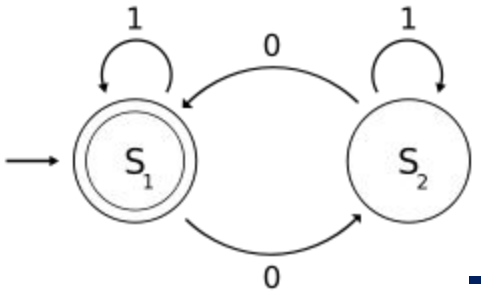


Intuitively, A is No Harder than B

Theorem. If $A \leq_p B$ and B has a polynomial time algorithm, then so does A.

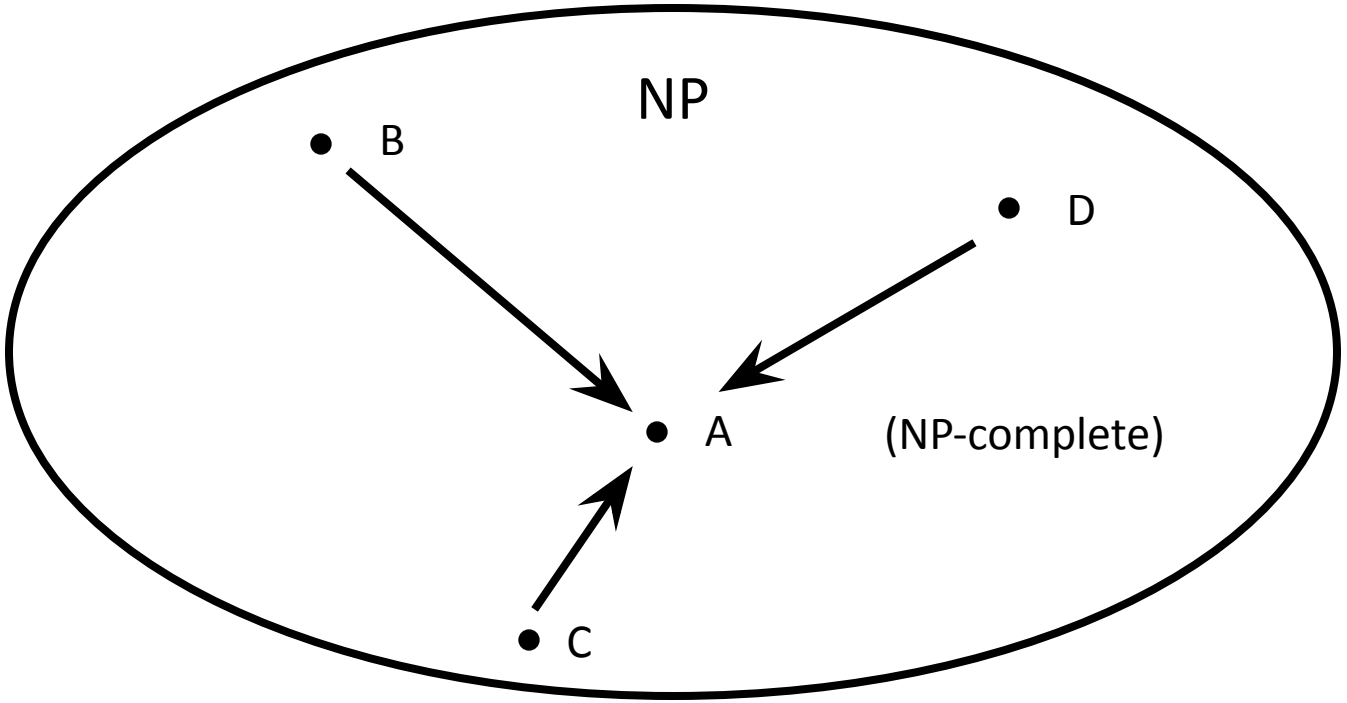
Proof.

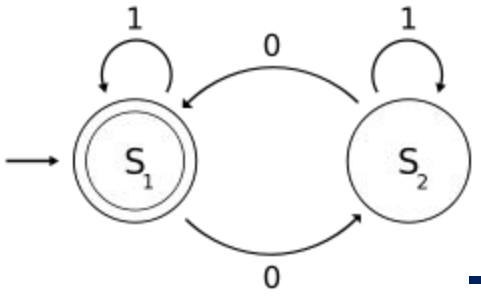




NP's Hardest Problems

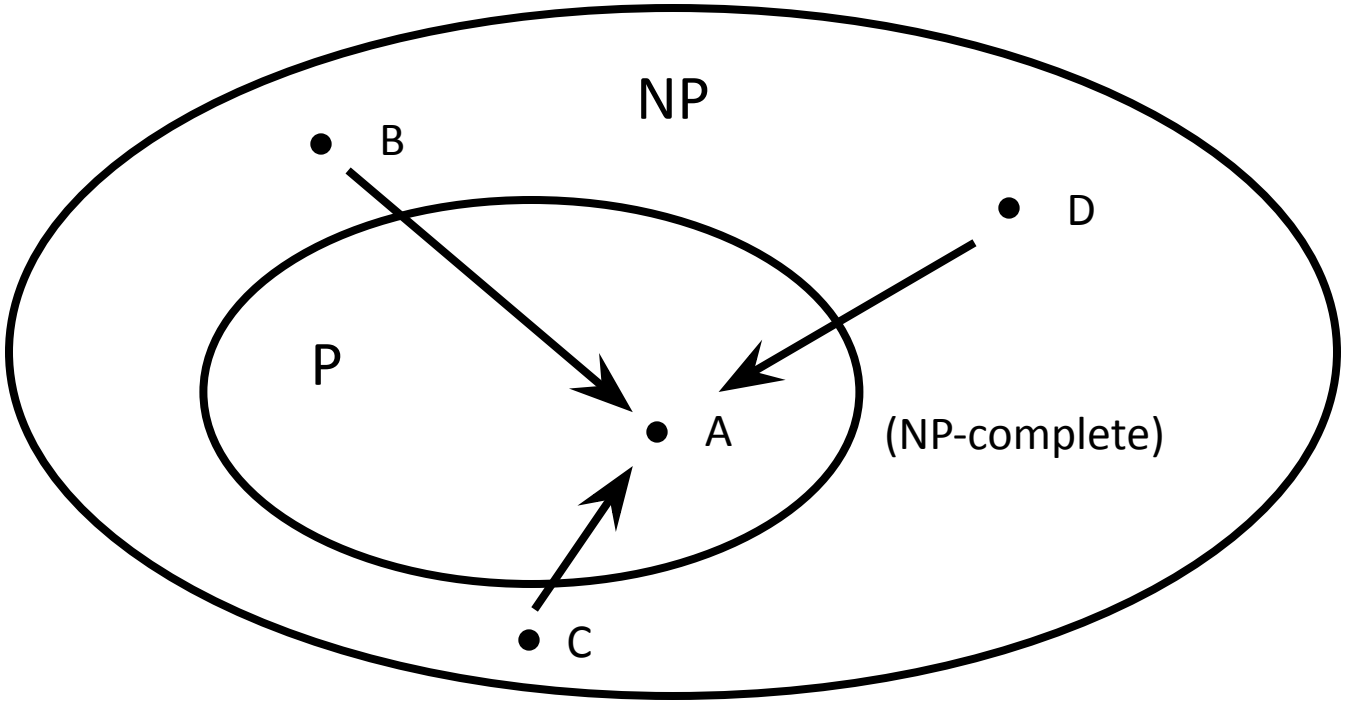
Definition. The set $A \in \text{NP}$ is *NP-complete* if for all $B \in \text{NP}$, $B \leq_p A$.

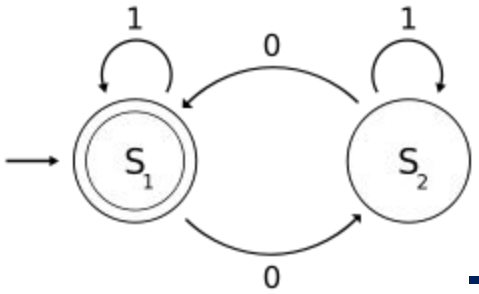




P = NP?

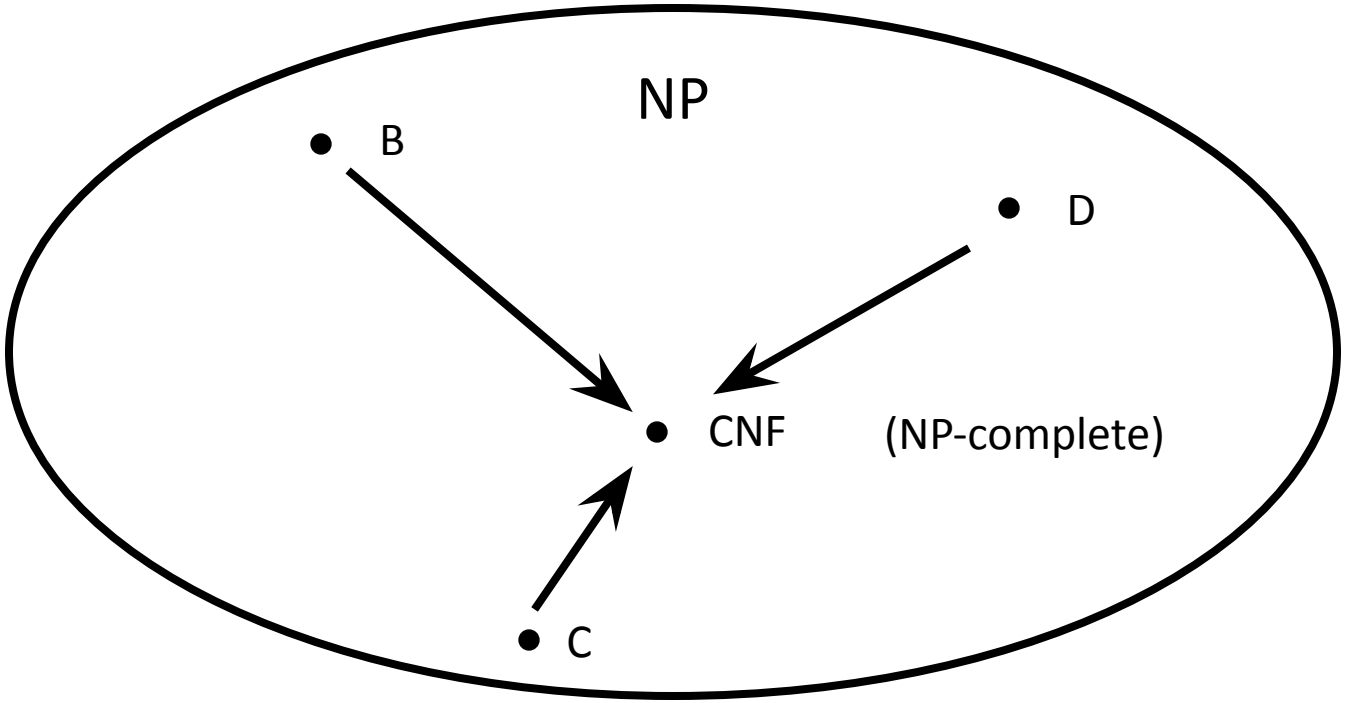
Theorem. If A is NP-complete, then $A \in$ P if and only if P = NP.

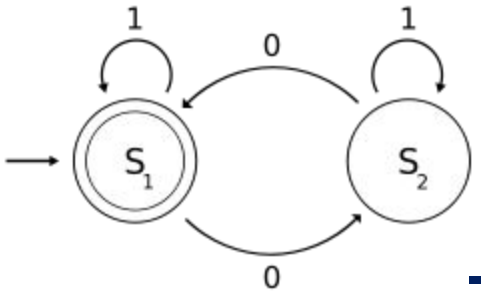




Cook's Theorem

Theorem. If $A \in$ NP then $A \leq_p$ CNFSat.

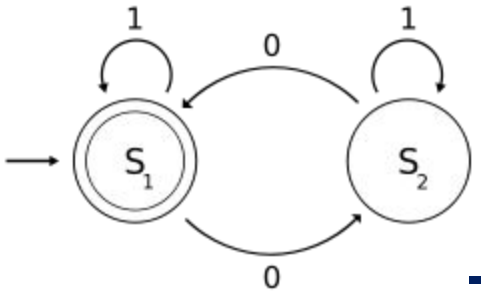




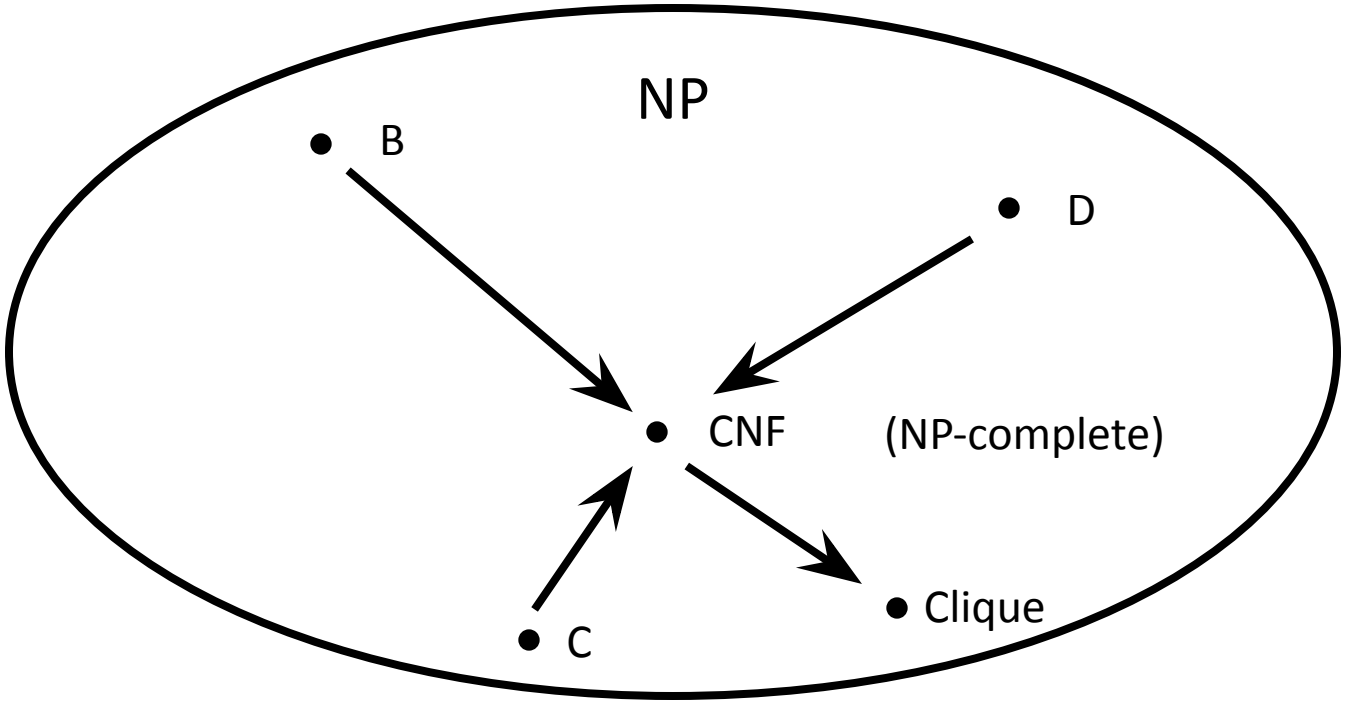
So What?

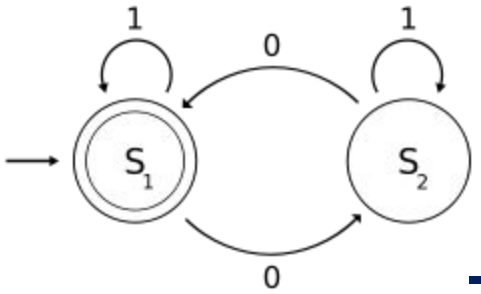
- The existence of one “natural” NP-complete problem is an interesting fact for the computer scientist.

- The existence of thousands of “natural” NP-complete problems is an essential fact for the computer scientist.



Clique is NP-Complete





NP-Complete Problems

DOUBLE-SAT = { <φ> | φ has at least two satisfying assignments }

Show *DOUBLE-SAT* is in NP.
Show *DOUBLE-SAT* is NP-complete.
HINT: Reduce *SAT* to *DOUBLE-SAT*. Create a new Boolean formula φ' based on Boolean formula φ such that φ is in *SAT* iff φ' is in *DOUBLE-SAT*.