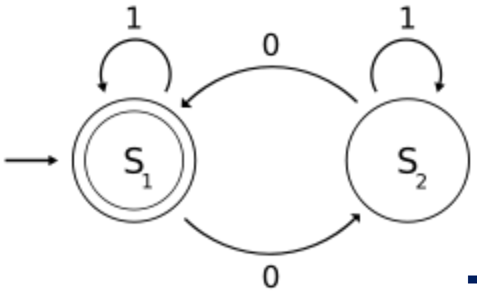
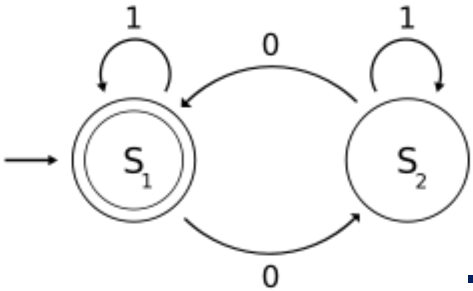
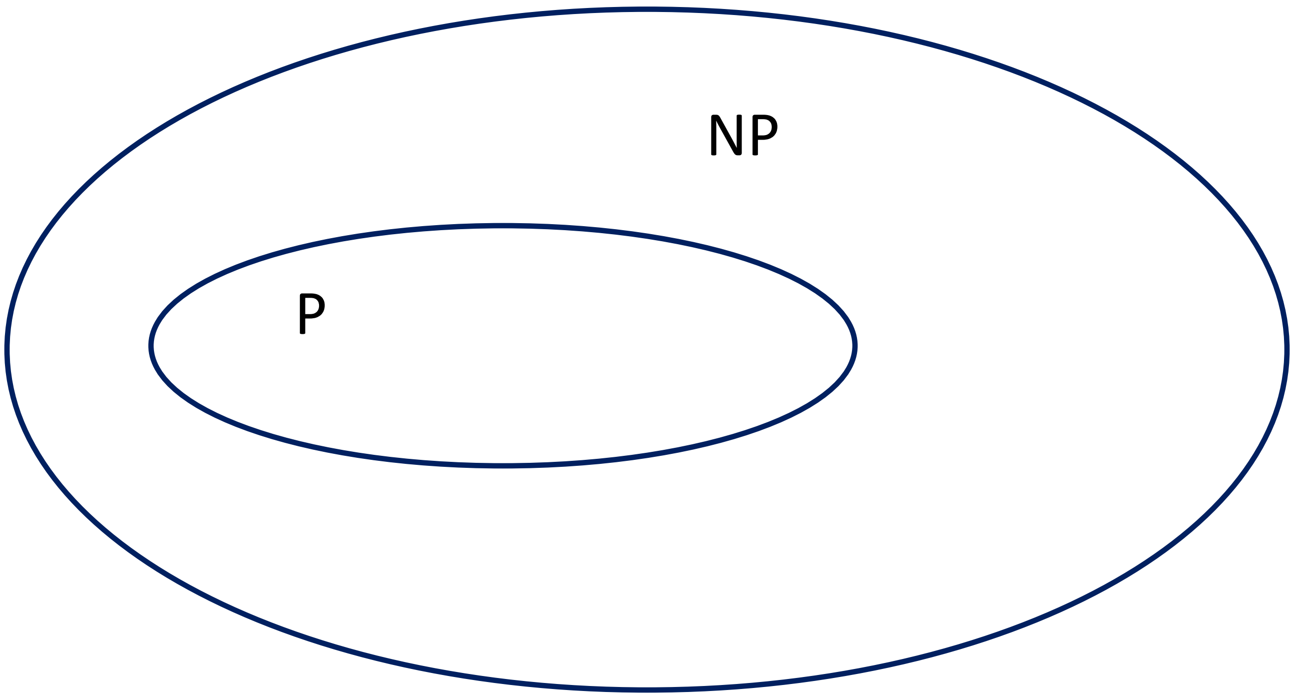


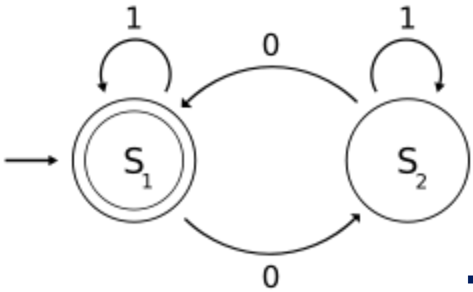
The Hardest Problem In The World

Sipser: Section 7.4 pages 299 – 311



The Classes P and NP?

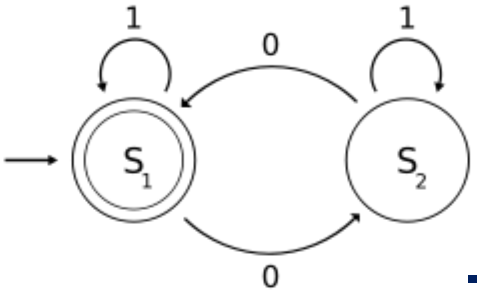




A Famous NP Problem

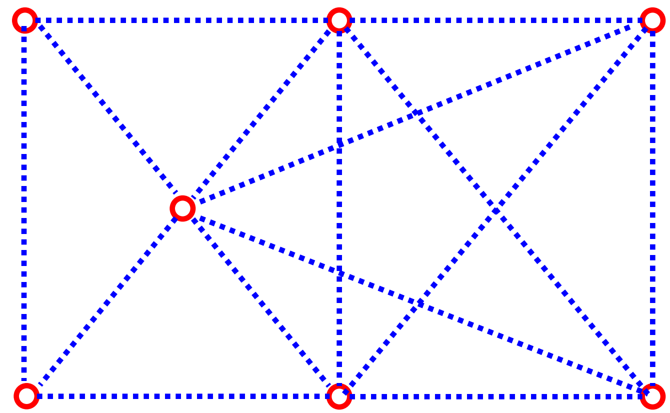
CNF satisfiability (*CNFSat*): given a Boolean formula *B* in conjunctive normal form (CNF), is there a truth assignment that satisfies *B*?

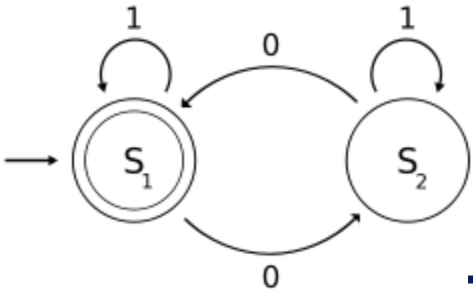
$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee \overline{x_2})$$



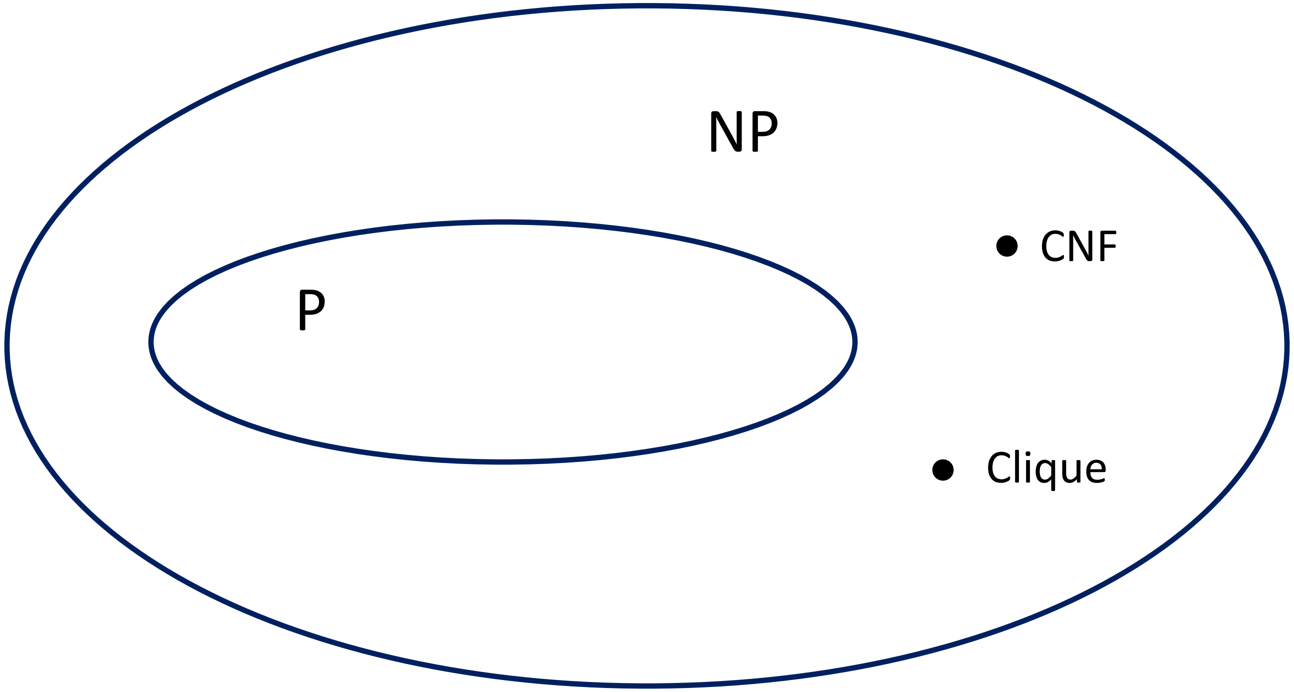
A Graph Theory NP Problem

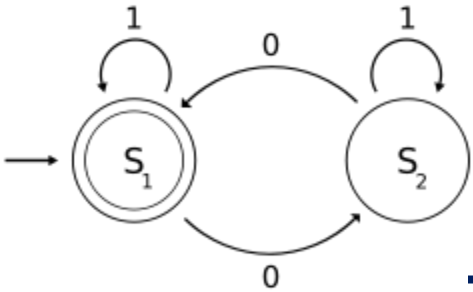
CLIQUE: given a graph $G = (V, E)$ and an integer k, does G contain C_k as a subgraph?





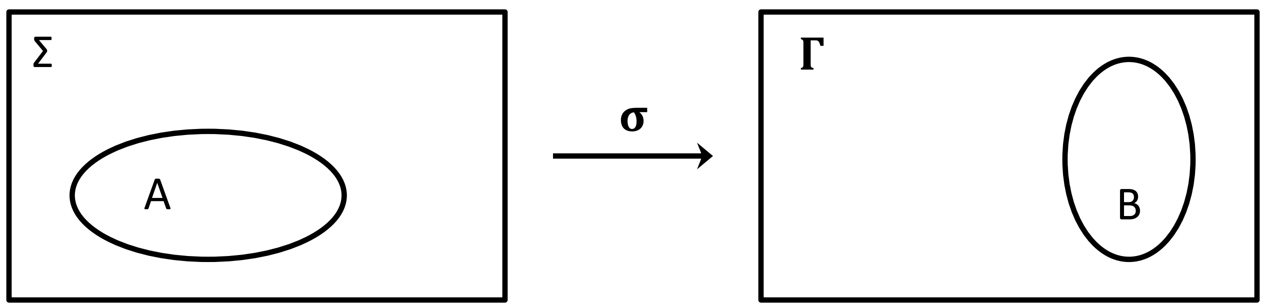
Which Problem is Harder?





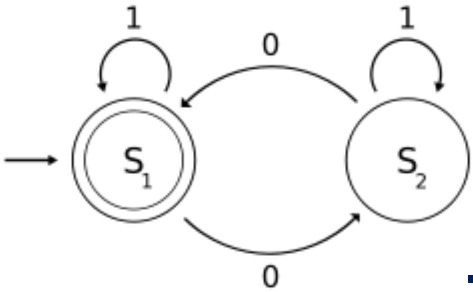
Polynomial Time Reduction

Definition. Let $A \subseteq \Sigma$ and $B \subseteq \Gamma$ be decision problems.



We write $A \leq_p B$ and say that *A reduces to B in polynomial time* if there is a polynomial time computable function $\sigma: \Sigma \longrightarrow \Gamma$ such that for all problem instances $x \in \Sigma$,

$$x \in A \text{ iff } \sigma(x) \in B.$$



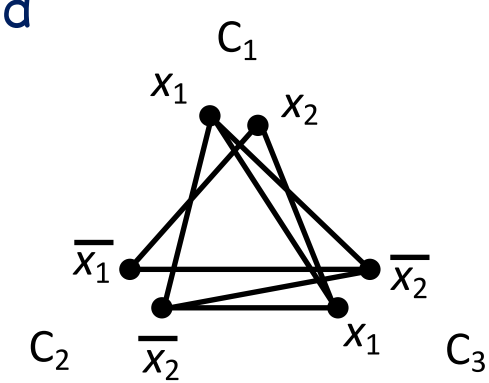
CNF $\leq_p$ Clique

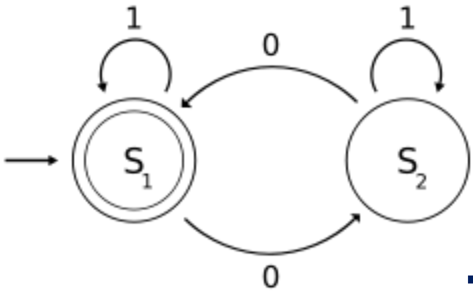
- Given a Boolean formula B in CNF, we show how to construct a graph G and an integer k such that G has a clique of size k iff B is satisfiable.

- Given

$$\underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2})}_{C_2} \wedge \underbrace{(x_1 \vee \overline{x_2})}_{C_3}$$

the construction would yield





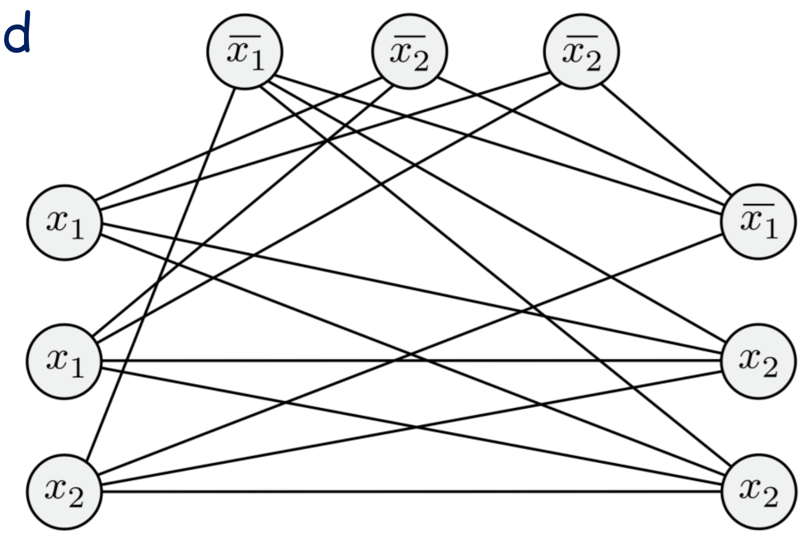
CNF $\leq_p$ Clique

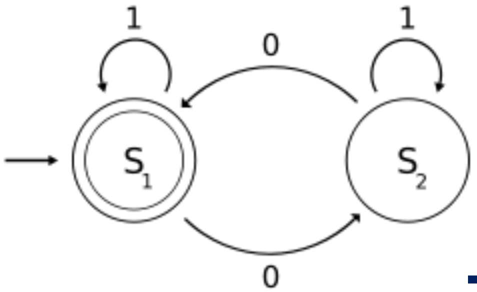
- Given a Boolean formula *B* in CNF, we show how to construct a graph *G* and an integer *k* such that *G* has a clique of size *k* iff *B* is satisfiable.

- Given

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

the construction would yield

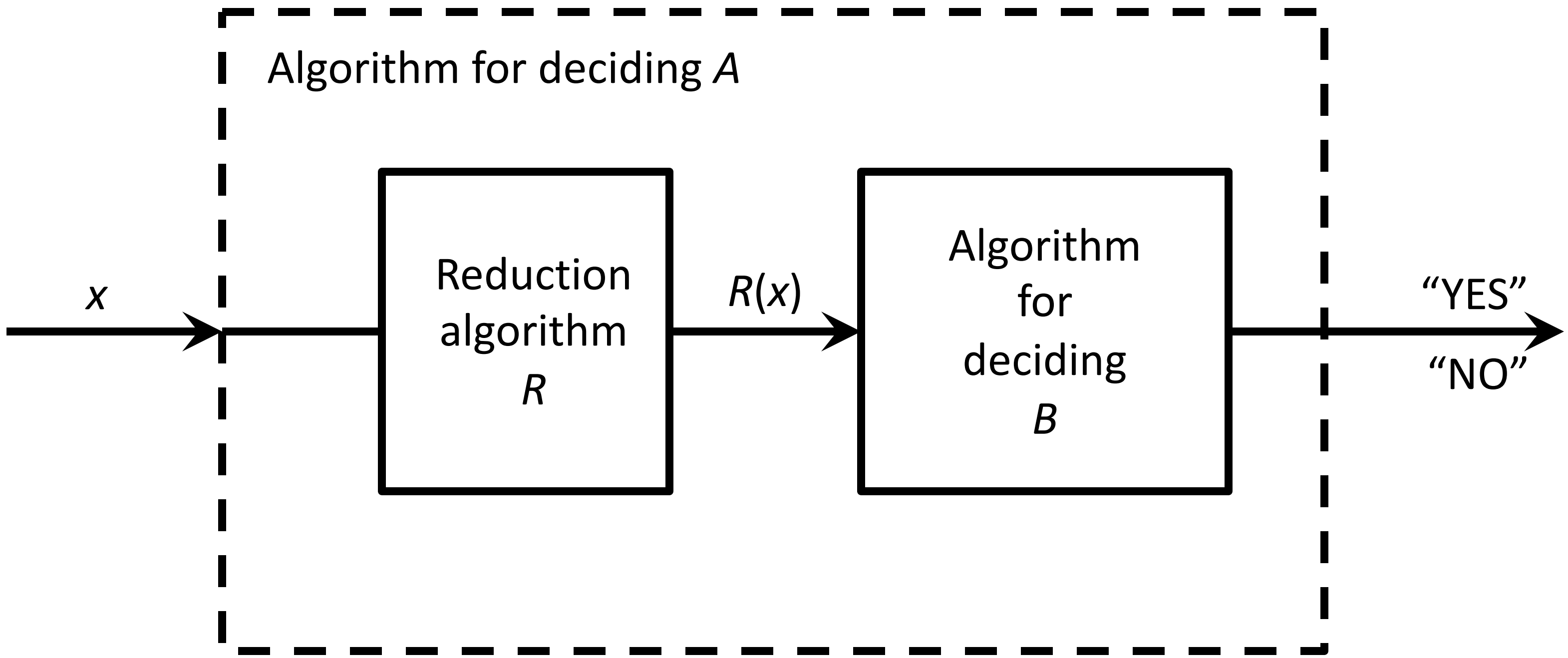


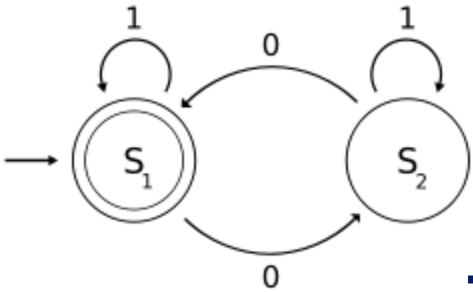


Intuitively, *A* is No Harder than *B*

Theorem. If $A \leq_p B$ and B has a polynomial time algorithm, then so does A.

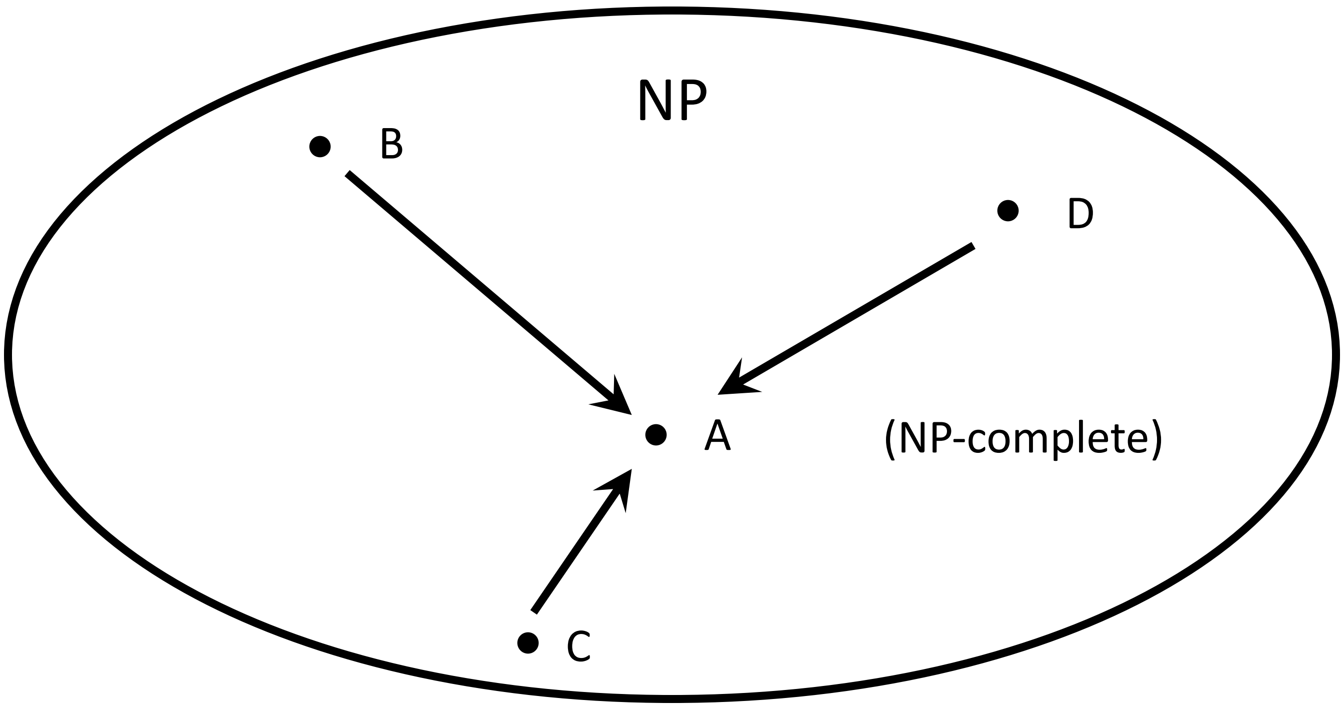
Proof.

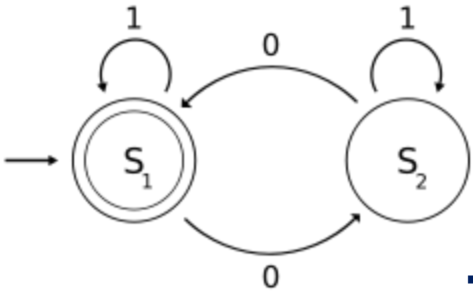




NP's Hardest Problems

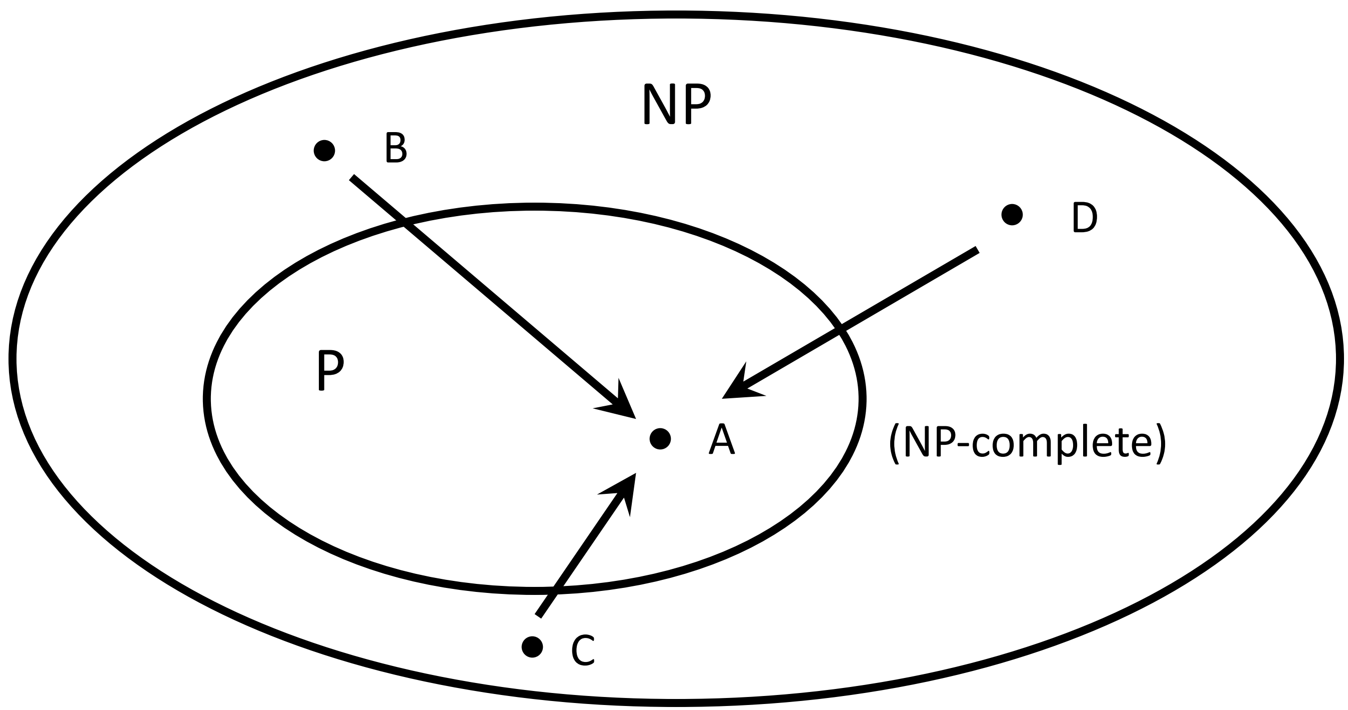
Definition. The set $A \in$ NP is *NP-complete* if for all $B \in$ NP, $B \leq_p A$.

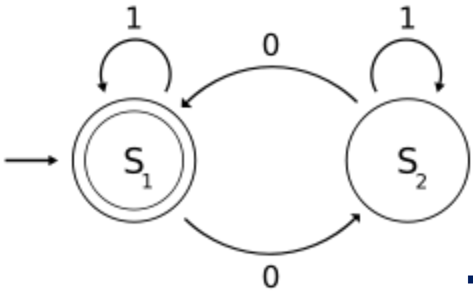




P = NP?

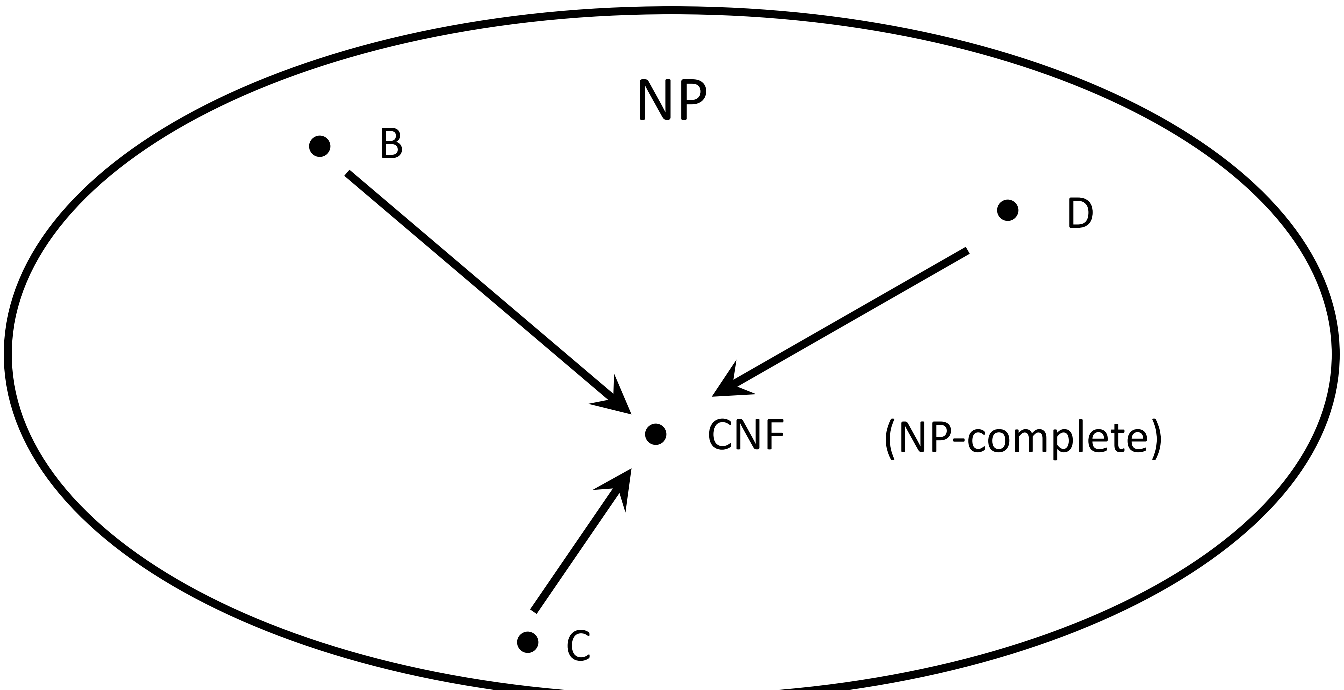
Theorem. If A is NP-complete, then $A \in P$ if and only if P = NP.

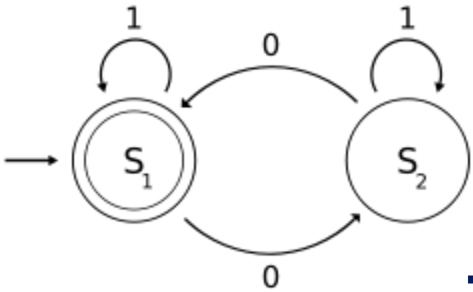




Cook's Theorem

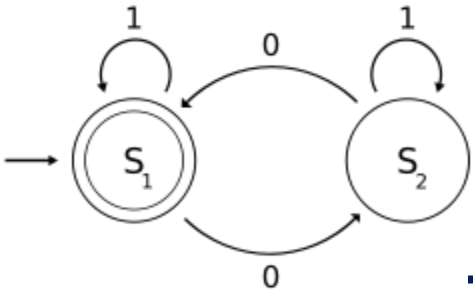
Theorem. If $A \in$ NP then $A \leq_p$ CNFSat.



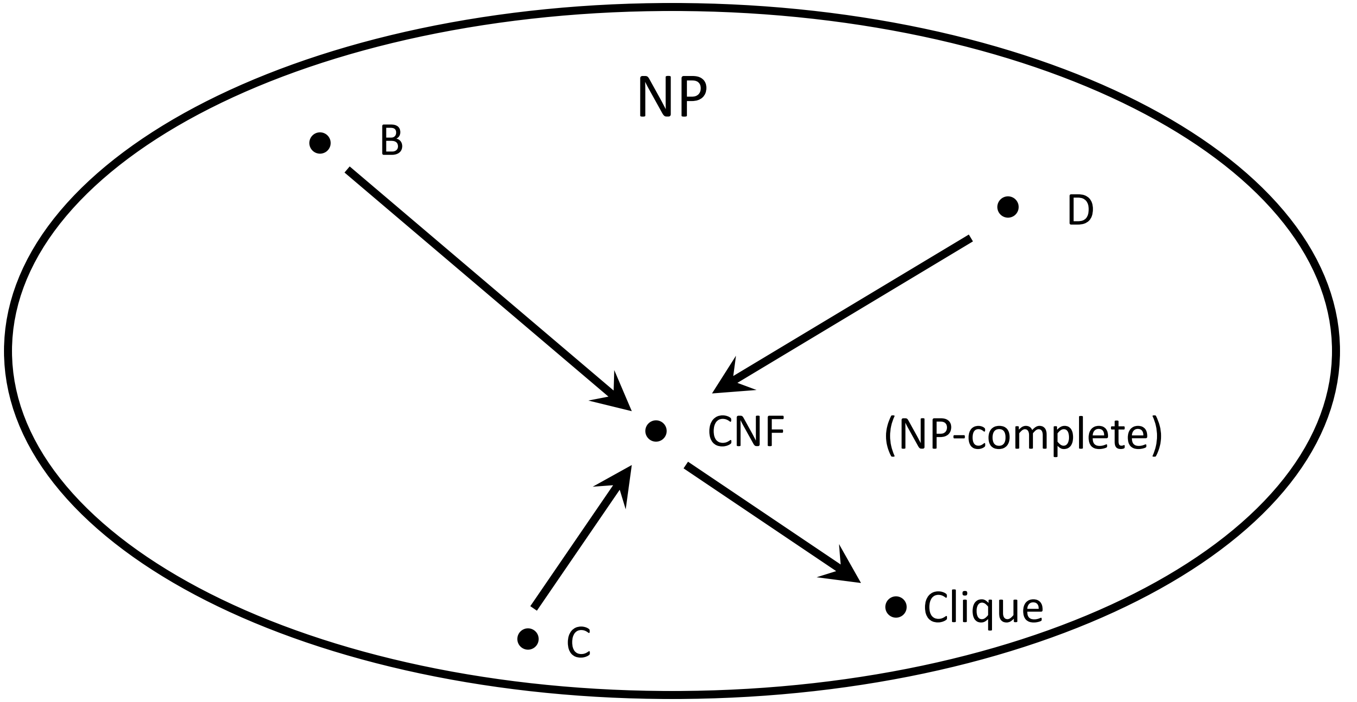


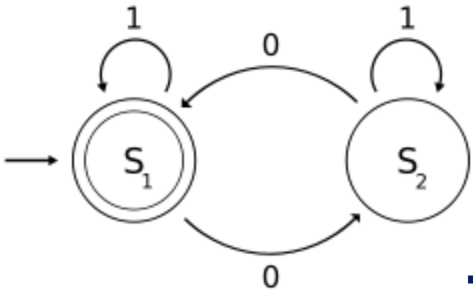
So What?

- The existence of one “natural” NP-complete problem is an interesting fact for the computer scientist.
- The existence of thousands of “natural” NP-complete problems is an essential fact for the computer scientist.



Clique is NP-Complete





NP-Complete Problems

DOUBLE-SAT = { $\langle\varphi\rangle$ | φ has at least two satisfying assignments }

Show *DOUBLE-SAT* is in NP. Show *DOUBLE-SAT* is NP-complete. HINT: Reduce *SAT* to *DOUBLE-SAT*. Create a new Boolean formula φ' based on Boolean formula φ such that φ is in *SAT* iff φ' is in *DOUBLE-SAT*.

HALF-CLIQUE = { $\langle G\rangle$ | *G* has a clique of size *m*/2 where *m* is the number of nodes in *G* }

Show *HALF-CLIQUE* is in NP. Show *HALF-CLIQUE* is NP-complete. HINT: Reduce *CLIQUE* to *HALF-CLIQUE*. Create a graph *H* such that $\langle G,k\rangle$ is in *CLIQUE* iff $\langle H\rangle$ is in *HALF-CLIQUE*. Consider the three cases when $k=m/2$, $k > m/2$, and $k < m/2$.