Regular Expressions
Consider the following NFA $N = (Q, \Sigma, \delta, q_1, F)$ where

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{a, b\}$
- $q_1$ is the start state
- $F = \{q_3\}$
- $\delta(q_1, a) = \{q_2, q_3\}$
- $\delta(q_1, b) = \emptyset$
- $\delta(q_2, a) = \emptyset$
- $\delta(q_2, b) = \{q_2\}$
- $\delta(q_3, a) = \{q_1, q_2\}$
- $\delta(q_3, b) = \{q_2, q_3\}$
- $\delta(q, \varepsilon) = \emptyset$ for all $q \in Q$
Regular Expressions

\[(0 \cup 1)0^*\]

\[(0 \cup 1)^*\]

\[\Sigma^*1\]

\[R^+\]
Long Ago in a Place Not Far Away
Old Home Week
Definition. Say that $R$ is a **regular expression** if $R$ is

1. $a$ for some $a$ in the alphabet $\Sigma$,
2. $\varepsilon$,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where $R_1$ and $R_2$ are regular expressions,
5. $(R_1 \circ R_2)$, where $R_1$ and $R_2$ are regular expressions,
6. $(R_1)^*$, where $R_1$ is a regular expression.
Working with Regular Expressions

\[ 0^*10^* = \{ w \mid \} \]

\[ = \{ w \mid w \text{ is a string of odd length} \} \]

\[ (0 \cup \varepsilon)(1 \cup \varepsilon) = \]

\[ (01)^* \emptyset = \]

\[ (+ \cup - \cup \varepsilon)(DD^* \cup DD^*D^* \cup D^*.DD^*) = \]

\[ \text{where } D = \{0,1,2,3,4,5,6,7,8,9\} \]
Let $R$ be a regular expression.

- $R \cup \emptyset =$
- $R \circ \varepsilon =$
- $R \cup \varepsilon =$
- $R \circ \emptyset =$
Theorem. A language is regular if and only if some regular expression describes it.

Proof. \((\Leftarrow)\)

1. If \(a \in \Sigma\), then \(a\) is regular.
2. \(\varepsilon\) is regular.
3. \(\emptyset\) is regular.
4. If \(R_1\) and \(R_2\) are regular, then \((R_1 \cup R_2)\) is regular.
5. If \(R_1\) and \(R_2\) are regular, then \((R_1 \circ R_2)\) is regular.
6. If \(R_1\) is regular, then \((R_1)^*\) is regular.
Proof in Action

Build an NFA that recognizes the regular expression: $(ab \cup a)^*$
Proof in Action

Build an NFA that recognizes the regular expression:

$$a(a \cup b)^*a$$