Wellesley CS 240

Integer Representation

Representation of integers: unsigned and signed

Modular arithmetic and overflow

Sign extension

Shifting and arithmetic

Multiplication

Casting

Fixed-width integer encodings

Unsigned $\subset \mathbb{N}$ **non-negative integers** only

Signed $\subset \mathbb{Z}$ both **negative** and **non-negative integers**

***n* bits** offer only **2^n distinct values.**

Terminology:

"Most-significant" bit(s)
or "high-order" bit(s)

"Least-significant" bit(s)
or "low-order" bit(s)

MSB 0110010110101001 **LSB**

(4-bit) **unsigned integer representation**

1	0	1	1
8	4	2	1
2^3	2^2	2^1	2^0
3	2	1	0

$= 1 \text{ x } 2^3 + 0 \text{ x } 2^2 + 1 \text{ x } 2^1 + 1 \text{ x } 2^0$

weight

position

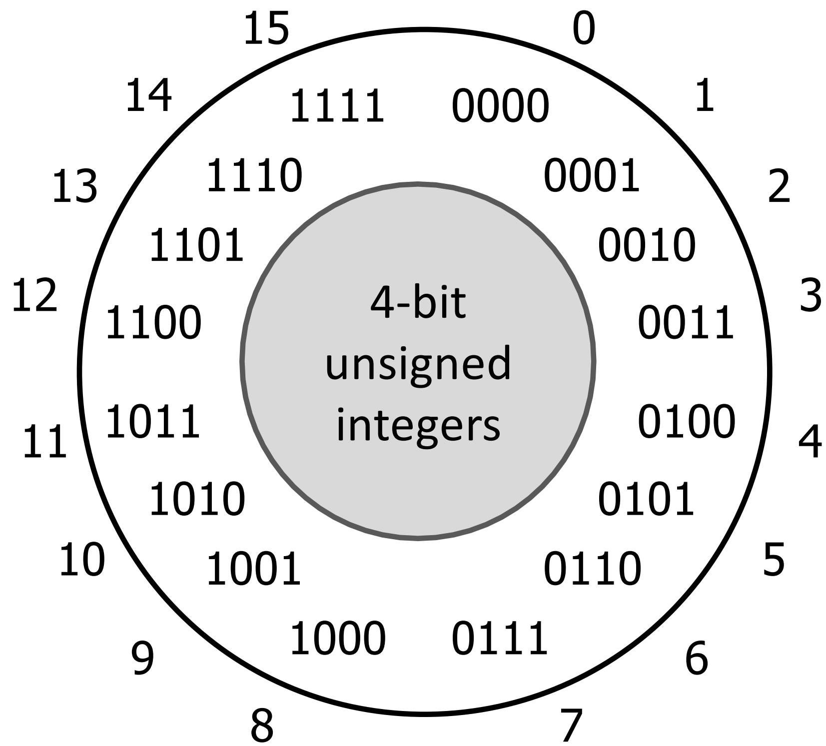
***n*-bit unsigned integers:**

minimum =

maximum =

modular arithmetic, overflow

$$\begin{array}{r} 11 \\ +\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 1011 \\ +\ 0010 \\ \hline \end{array}$$



$$\begin{array}{r} 13 \\ +\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 1101 \\ +\ 0101 \\ \hline \end{array}$$

x+y in *n*-bit unsigned arithmetic is in math

unsigned overflow =

=

Unsigned addition *overflows* if and only if

sign-magnitude

!!!

Most-significant bit (MSB) is ***sign bit***

0 means non-negative 1 means negative

Remaining bits are an unsigned magnitude

8-bit sign-magnitude:

00000000 represents _____

01111111 represents _____

10000101 represents _____

10000000 represents _____

ex

Anything weird here?

Arithmetic?

Example:

4 - 3 != 4 + (-3)

$$\begin{array}{r} 00000100 \\ +\underline{10000011} \end{array}$$

Zero?

(4-bit) **two's complement signed integer representation**

compare to unsigned

1	0	1	1
-2^3	2^2	2^1	2^0

$= 1 \times -2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

***4*-bit two's complement integers:**

minimum =

maximum =

two's complement vs. **unsigned**

_	_	… _	_	_
2^{n-1}	2^{n-2}	… 2^2	2^1	2^0
$\mathbf{-2^{n-1}}$	2^{n-2}	… 2^2	2^1	2^0

unsigned places

two's complement places

What's the difference?

n-bit minimum =

n-bit maximum =

8-bit representations

ex

0 0 0 0 1 0 0 1

1 0 0 0 0 0 0 1

1 1 1 1 1 1 1 1

0 0 1 0 0 1 1 1

n-bit two's complement numbers:

minimum =

maximum =

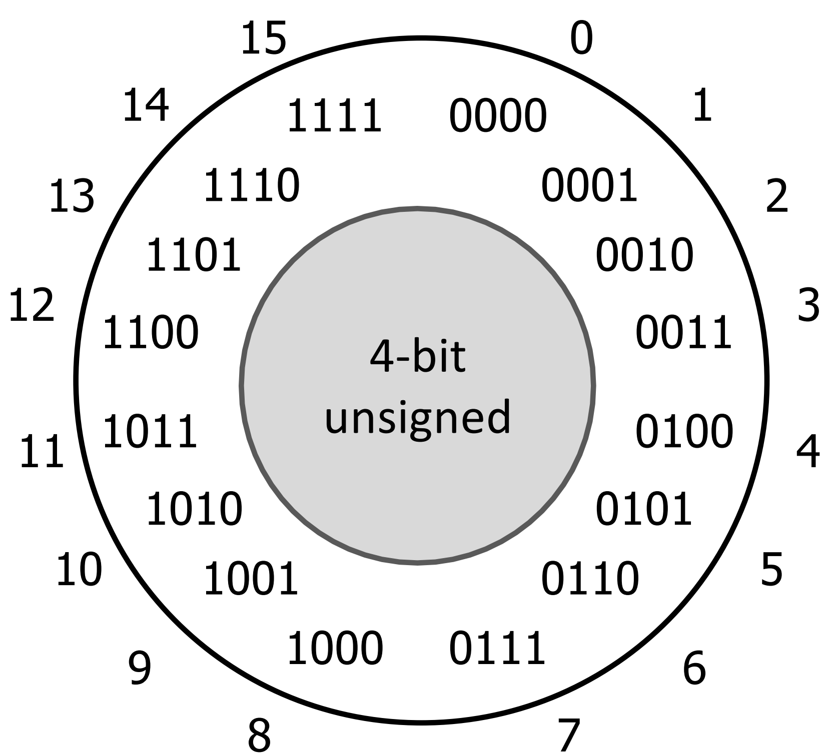
4-bit **unsigned** vs. 4-bit **two's complement**

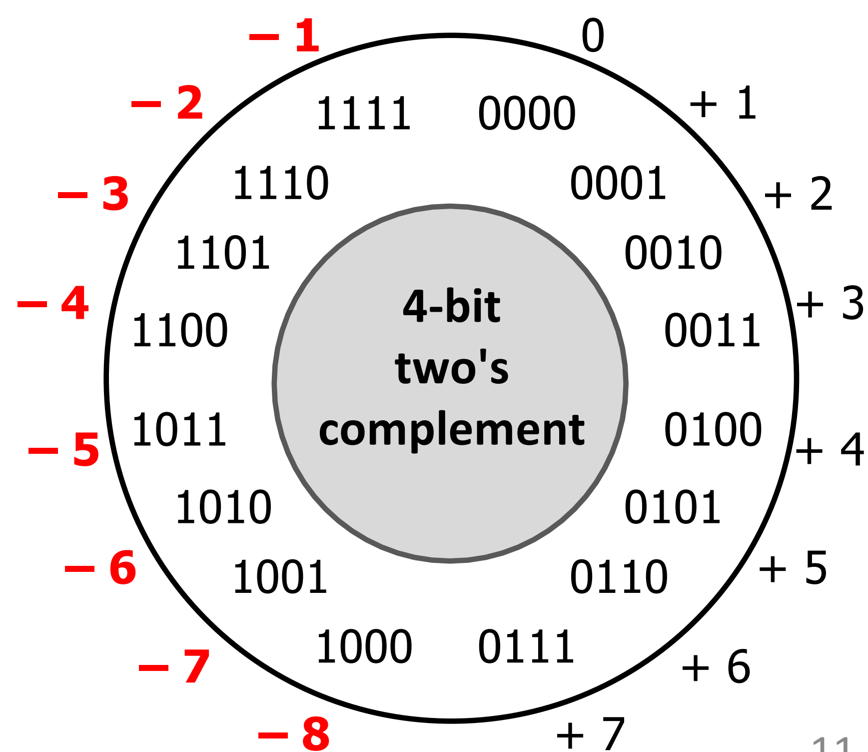
1 0 1 1

$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$1 \times -2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

11 **difference = ___ = 2__** -5

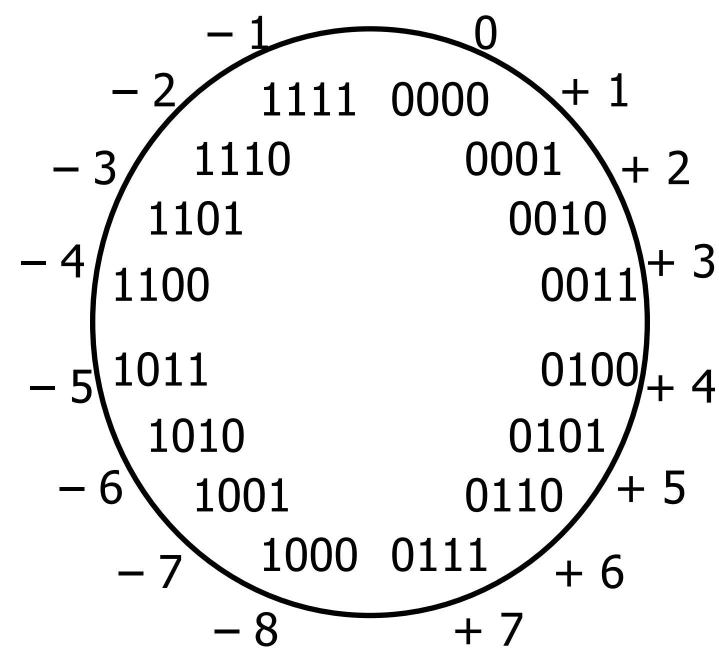




two's complement **addition**

$$\begin{array}{r} 2 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 0010 \\ +\ 0011 \\ \hline \end{array} \qquad\qquad \begin{array}{r} -2 \\ +\ -3 \\ \hline \end{array} \qquad \begin{array}{r} 1110 \\ +\ 1101 \\ \hline \end{array}$$

$$\begin{array}{r} -2 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 1110 \\ +\ 0011 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 2 \\ +\ -3 \\ \hline \end{array} \qquad \begin{array}{r} 0010 \\ +\ 1101 \\ \hline \end{array}$$



Modular Arithmetic

two's complement ***overflow***

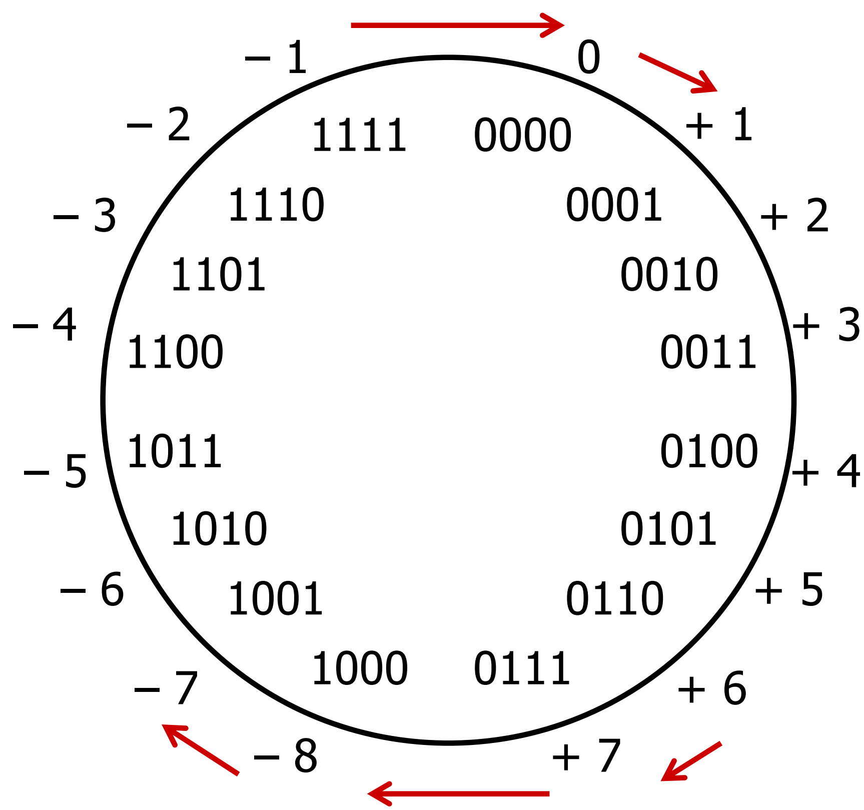
Addition *overflows*

if and only if

if and only if

$$\begin{array}{r} -1 \\ +\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 1111 \\ +\ 0010 \\ \hline \end{array}$$

$$\begin{array}{r} 6 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 0110 \\ +\ 0011 \\ \hline \end{array}$$



Modular Arithmetic

Some CPUs/languages raise exceptions on overflow.
C and Java cruise along silently... Feature? Oops?

Reliability

Ariane 5 Rocket, 1996

Exploded due to **cast** of
64-bit floating-point number
to 16-bit signed number.
Overflow.

Boeing 787, 2015

"... a **Model 787 airplane** … can lose all alternating current (AC) electrical power … caused by a **software counter** internal to the GCUs that will **overflow** after **248 days** of continuous power. We are issuing this AD to prevent loss of all AC electrical power, which could result in **loss of control of the airplane**."
--FAA, April 2015

A few reasons two's complement is awesome

Addition, subtraction, hardware

Sign

Negative one

Complement rules

Another derivation

ex

How should we represent 8-bit negatives?

- For all positive integers ***x***,
  we want the representations of ***x*** and ***–x*** to sum to zero.
- We want to use the standard addition algorithm.

```
  00000001        00000010        00000011
+               +               +
  --------        --------        --------
  00000000        00000000        00000000
```

- Find a rule to represent –x where that works…

Convert/cast *signed number to* ***larger*** *type.*

0 0 0 0 0 0 1 0 8-bit 2

_ _ _ _ _ _ _ _ 0 0 0 0 0 0 1 0 16-bit 2

1 1 1 1 1 1 0 0 8-bit -4

_ _ _ _ _ _ _ _ 1 1 1 1 1 1 0 0 16-bit -4

Rule/name?

unsigned **shifting** and **arithmetic**

unsigned
x = 27;

y = x << 2;

y == 108

0 0 0 1 1 0 1 1

0 0 0 1 1 0 1 1 0 0

logical shift left

logical shift right

1 1 1 0 1 1 0 1

0 0 1 1 1 0 1 1 0 1

unsigned
x = 237;

y = x >> 2;

y == 59

two's complement **shifting** and **arithmetic**

signed

x = -101;

1 0 0 1 1 0 1 1

y = x << 2;

y == 108

1 0 0 1 1 0 1 1 0 0

logical shift left

arithmetic shift right

1 1 1 0 1 1 0 1

signed

x = -19;

y = x >> 2;

1 1 1 1 1 0 1 1 0 1

y == -5

shift-*and*-***add***

ex

Available operations

`x << k` implements `x *` 2^k

`x + y`

Implement `y = x * 24` using only <<, +, and integer literals

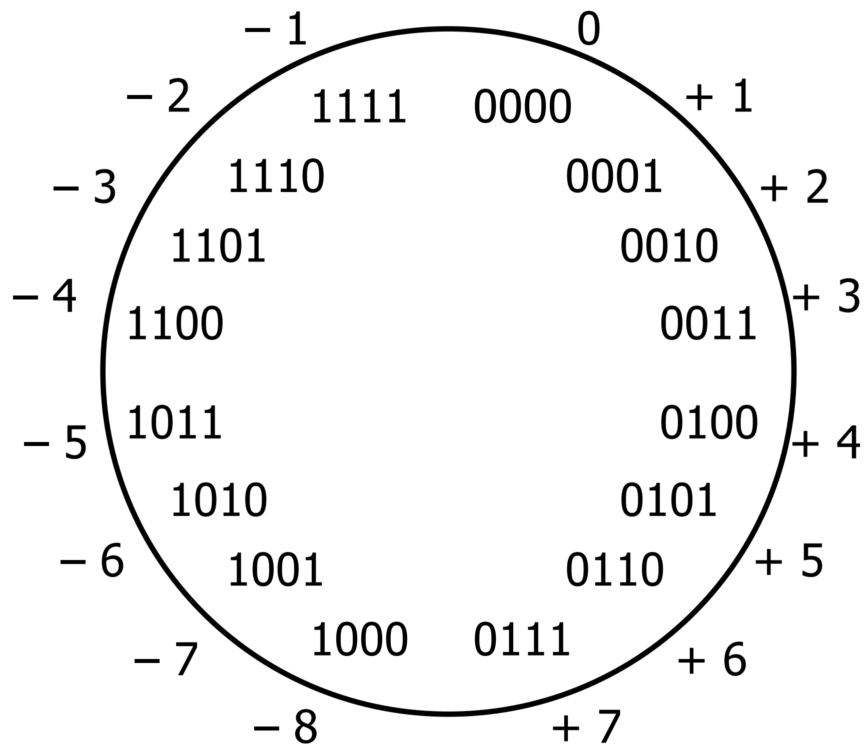
What does this function compute?

ex

```
unsigned puzzle(unsigned x, unsigned y) {
  unsigned result = 0;
  for (unsigned i = 0; i < 32; i++){
    if (y & (1 << i)) {
      result = result + (x << i);
    }
  }
  return result;
}
```

multiplication

2	0010
x 3	x 0011
6	00000100
-2	1110
x 2	x 0010
-4	11111100

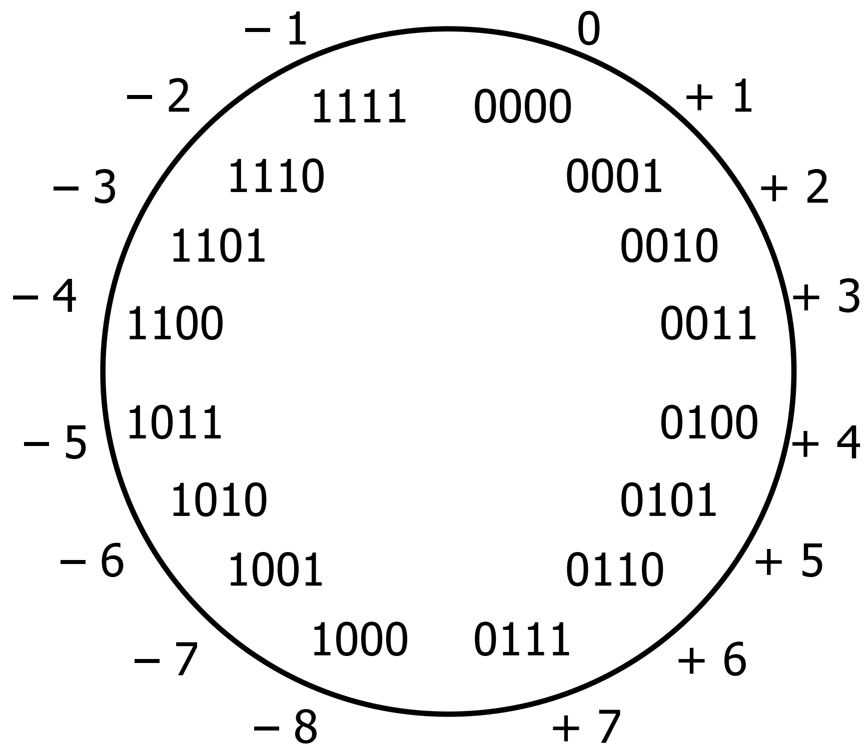


Modular Arithmetic

multiplication

$$\begin{array}{r} 5 \\ \text{x } 4 \\ \hline \cancel{20} \\ 4 \end{array} \qquad \begin{array}{r} 0101 \\ \text{x } 0100 \\ \hline 00010100 \end{array}$$

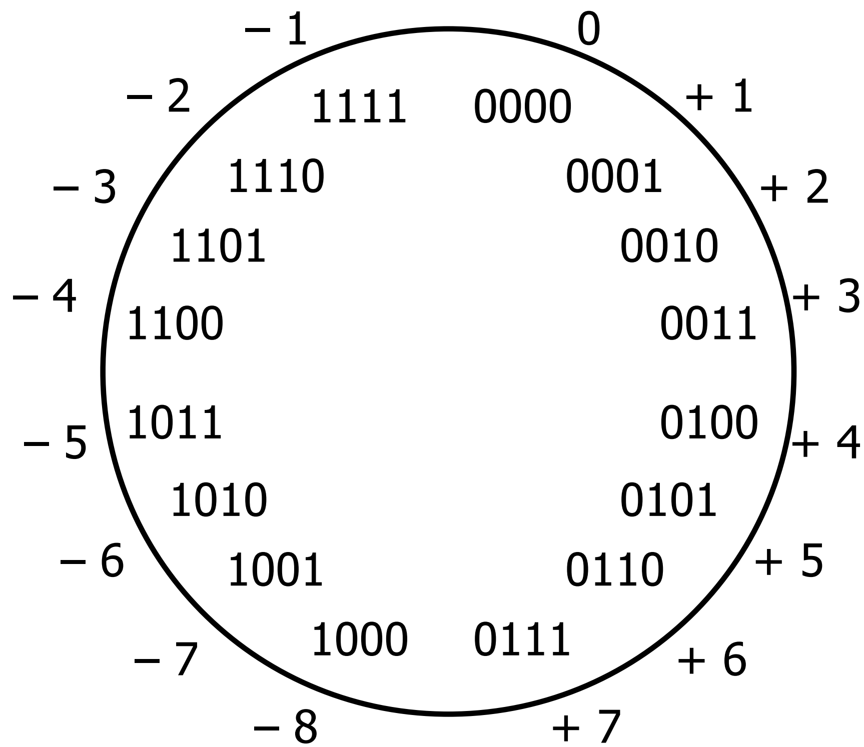
$$\begin{array}{r} -3 \\ \text{x } 7 \\ \hline \cancel{-21} \\ -2 \end{array} \qquad \begin{array}{r} 1101 \\ \text{x } 0111 \\ \hline 11101011 \end{array}$$



Modular Arithmetic

multiplication

5	0101
x 5	x 0101
~~25~~	00011001
-7	
-2	1110
x 6	x 0110
~~-12~~	11110100
4	



Modular Arithmetic

Casting Integers in C

!!!

Number literals: `37` is signed, `37U` is unsigned

Integer Casting: *bits unchanged, just reinterpreted.*

Explicit casting:

```
int tx = (int) 73U;              // still 73
unsigned uy = (unsigned) -4;     // big positive #
```

Implicit casting: **Actually does**

```
tx = ux;                  // tx = (int)ux;
uy = ty;                  // uy = (unsigned)ty;
void foo(int z) { ... }
foo(ux);                  // foo((int)ux);
if (tx < ux) ...          // if ((unsigned)tx < ux) ...
```

More Implicit Casting in C

!!!

If you **mix unsigned** and **signed** in a single expression, then
signed** values are **implicitly cast to unsigned.

How are the argument bits interpreted?

Argument$_1$	Op	Argument$_2$	Type	Result
`0`	`==`	`0U`	unsigned	1
`-1`	`<`	`0`	signed	1
`-1`	`<`	`0U`	unsigned	**0**
`2147483647`	`<`	`-2147483648`		
`2147483647U`	`<`	`-2147483648`		
`-1`	`<`	`-2`		
`(unsigned)-1`	`<`	`-2`		
`2147483647`	`<`	`2147483648U`		
`2147483647`	`<`	`(int)2147483648U`		

Note: T_{min} = -2,147,483,648 T_{max} = 2,147,483,647