CS 240
Foundations of Computer Systems

Integer Representation

Representation of integers: unsigned and signed
Modular arithmetic and overflow
Sign extension
Shifting and arithmetic
Multiplication
Casting

Fixed-width integer encodings

Unsigned $\subset \mathbb{N}$ non-negative integers only

Signed $\subset \mathbb{Z}$ both negative and non-negative integers

n bits offer only 2^n distinct values.

Terminology:

"Most-significant" bit(s) or "high-order" bit(s)

"Least-significant" bit(s) or "low-order" bit(s)

MSB 0110010110101001 **LSB**

(4-bit) **unsigned integer representation**

1	0	1	1
8	4	2	1
2^3	2^2	2^1	2^0
3	2	1	0

$= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

weight

position

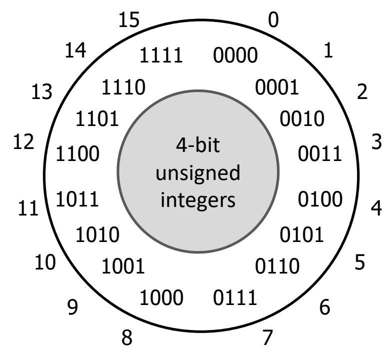
n-bit unsigned integers:

minimum =

maximum =

modular arithmetic, overflow

$$\begin{array}{r} 11 \\ \underline{+\ 2} \end{array} \qquad \begin{array}{r} 1 \\ 1011 \\ \underline{+\ 0010} \\ 1101 \end{array}$$



$$\begin{array}{r} 13 \\ \underline{+\ 5} \end{array} \qquad \begin{array}{r} 1\ 1\ \ 1\ \ \\ 1101 \\ \underline{+\ 0101} \\ 0010 \end{array}$$

`x+y` in n-bit unsigned arithmetic is in math

unsigned overflow =

=

Unsigned addition *overflows* if and only if

sign-magnitude

!!!

Most-significant bit (MSB) is *sign bit*

0 means non-negative1 means negative

Remaining bits are an unsigned magnitude

8-bit sign-magnitude:

00000000 represents _____

01111111 represents _____

10000101 represents _____

10000000 represents _____

Anything weird here?

Arithmetic?

Example:
4 - 3 != 4 + (-3)

```
 00000100
+10000011
```

Zero?

ex

(4-bit) **two's complement** **signed integer representation**

compare to unsigned

1	0	1	1
$-(2^3)$	2^2	2^1	2^0

$= 1 \times -(2^3) + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

4-bit two's complement integers:

minimum =

maximum =

two's complement vs. unsigned

_	_	… _	_	_
2^{n-1}	2^{n-2}	… 2^2	2^1	2^0
$-(2^{n-1})$	2^{n-2}	… 2^2	2^1	2^0

unsigned places

two's complement places

unsigned range (2^n values)

$-(2^{(n-1)})$ 0 $2^{(n-1)} - 1$ $2^n - 1$

two's complement range (2^n values)

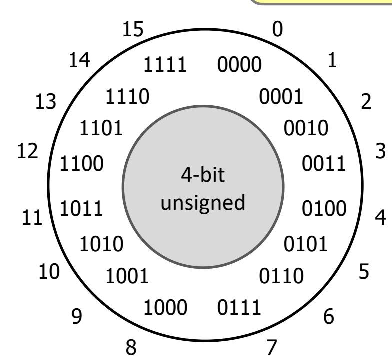
4-bit **unsigned** vs. 4-bit **two's complement**

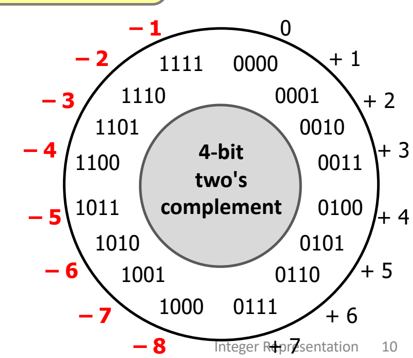
1 0 1 1

$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$1 \times -2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

11 ← **difference = ___ = 2—** → -5





8-bit representations

ex

0 0 0 0 1 0 0 1   1 0 0 0 0 0 0 1

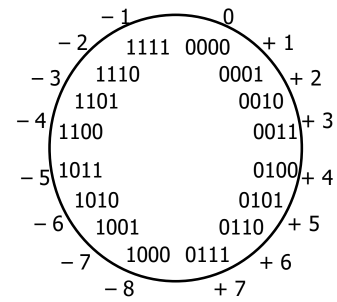
1 1 1 1 1 1 1 1   0 0 1 0 0 1 1 1

n-bit two's complement numbers:

minimum =   maximum =

two’s complement **addition**

2	0010	-2	1110
+ 3	+ 0011	+ -3	+ 1101
5		-5	
-2	1110	2	0010
+ 3	+ 0011	+ -3	+ 1101
1		-1	



Modular Arithmetic

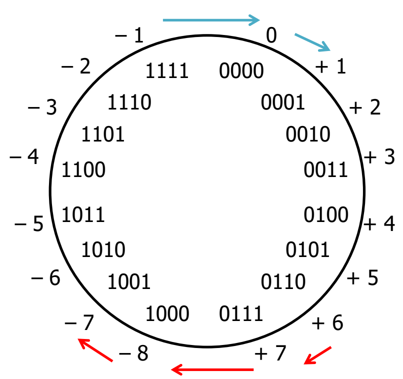
two’s complement ***overflow***

Addition *overflows*

if and only if

if and only if

-1	1111
+ 2	+ 0010
6	0110
+ 3	+ 0011



Modular Arithmetic

Some CPUs/languages raise exceptions on overflow.
C and Java cruise along silently... Feature? Oops?

Reliability

Ariane 5 Rocket, 1996

Exploded due to **cast** of 64-bit floating-point number to 16-bit signed number. **Overflow.**

Boeing 787, 2015

"... a **Model 787 airplane** ... can lose all alternating current (AC) electrical power ... caused by a **software counter** internal to the GCUs that will **overflow** after **248 days** of continuous power. We are issuing this AD to prevent loss of all AC electrical power, which could result in **loss of control of the airplane**." --FAA, April 2015

A few reasons two's complement is awesome

Arithmetic hardware

Sign

Negative one

Complement rules

Another derivation

ex

How should we represent 8-bit negatives?

- For all positive integers ***x***,
  we want the representations of ***x*** and ***–x*** to sum to zero.
- We want to use the standard addition algorithm.

```
 111111111     11111111     111111111
  00000001     00000010      00000011
+ 11111111   + 11111110    + 11111101
----------   ----------    ----------
  00000000     00000000      00000000
```

- Find a rule to represent –x where that works…

Convert/cast signed number to larger type.

0 0 0 0 0 0 1 0	8-bit 2
_ _ _ _ _ _ _ _ 0 0 0 0 0 0 1 0	16-bit 2
1 1 1 1 1 1 0 0	8-bit -4
_ _ _ _ _ _ _ _ 1 1 1 1 1 1 0 0	16-bit -4

Rule/name?

Sign extension for two's complement

0 0 0 0 0 0 1 0	8-bit 2
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0	16-bit 2
1 1 1 1 1 1 0 0	8-bit -4
1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0	16-bit -4

Casting from smaller to larger signed type does sign extension.

unsigned **shifting** and **arithmetic**

unsigned
x = 27;
y = x << 2;
y == 108

0 0 0 1 1 0 1 1

0 0 0 1 1 0 1 1 0 0

logical shift left

n = shift distance in bits, w = width of encoding in bits

logical shift right

1 1 1 0 1 1 0 1

0 0 1 1 1 0 1 1 0 1

unsigned
x = 237;
y = x >> 2;
y == 59

two's complement **shifting** and **arithmetic**

signed
x = -101;
y = x << 2;
y == 108

1 0 0 1 1 0 1 1

1 0 0 1 1 0 1 1 0 0

logical shift left

n = shift distance in bits, w = width of encoding in bits

arithmetic shift right

1 1 1 0 1 1 0 1

1 1 1 1 1 0 1 1 0 1

signed
x = -19;
y = x >> 2;
y == -5

shift**-and-**add

ex

Available operations

`x << k` implements `x *` ***2^k***

`x + y`

Implement `y = x * 24` using only <<, +, and integer literals

Parenthesize shifts to be clear about precedence, which may not always be what you expect.

What does this function compute?

ex

```
unsigned puzzle(unsigned x, unsigned y) {
  unsigned result = 0;
  for (unsigned i = 0; i < 32; i++){
    if (y & (1 << i)) {
      result = result + (x << i);
    }
  }
  return result;
}
```

See Bits assignment prep exercise.

What does this function compute?

ex

Downsize to fake unsigned nybble type (4 bits) to make this easier to write…

```
nybble puzzle(nybble x, nybble y) {
  nybble result = 0;
  for (nybble i = 0; i < 4; i++){
    if (y & (1 << i)) {
      result = result + (x << i);
    }
  }
  return result;
}
```

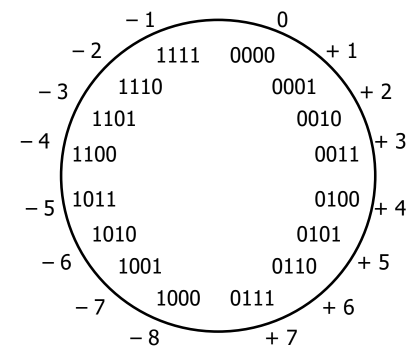
y_2	x_2

i_{10}	$y\&(1<<i)_2$	$result_2$
		0 0 0 0
0		
1		
2		
3		
4		

See Bits assignment prep exercise.

multiplication

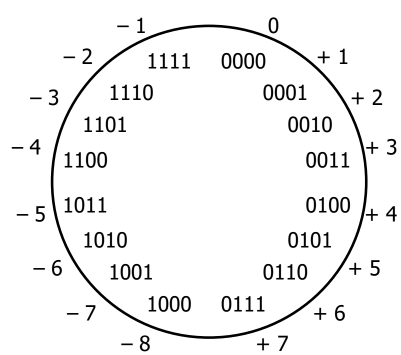
2	0010
x 3	x 0011
6	00000110
-2	1110
x 2	x 0010
-4	11111100



Modular Arithmetic

multiplication

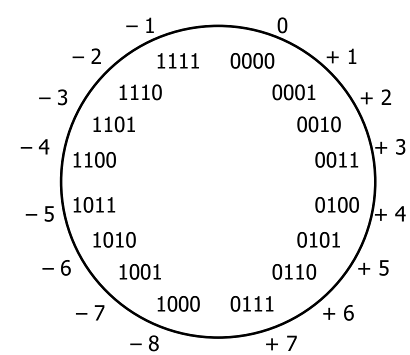
5	0101
x 4	x 0100
~~20~~	00010100
4	
-3	1101
x 7	x 0111
~~-21~~	11101011
-5	



Modular Arithmetic

multiplication

5	0101
x 5	x 0101
~~25~~	00011001
-7	
-2	1110
x 6	x 0110
~~-12~~	11110100
4	



Modular Arithmetic

Casting Integers in C

!!!

Number literals: `37` is signed, `37U` is unsigned

Integer Casting: *bits unchanged, just reinterpreted.*

Explicit casting:

```
int tx = (int) 73U;      // still 73
unsigned uy = (unsigned) -4;  // big positive #
```

Implicit casting: **Actually does**

```
tx = ux;     // tx = (int)ux;
uy = ty;     // uy = (unsigned)ty;
void foo(int z) { ... }
foo(ux);     // foo((int)ux);
if (tx < ux) ...   // if ((unsigned)tx < ux) ...
```


More Implicit Casting in C

!!!

If you mix unsigned and signed in a single expression, then
***signed** values are **implicitly cast to <u>unsigned</u>**.*

How are the argument bits interpreted?

Argument$_1$	Op	Argument$_2$	Type	Result
`0`	`==`	`0U`	unsigned	1
`-1`	`<`	`0`	signed	1
`-1`	`<`	`0U`	unsigned	0
`2147483647`	`<`	`-2147483647-1`		
`2147483647U`	`<`	`-2147483647-1`		
`-1`	`<`	`-2`		
`(unsigned)-1`	`<`	`-2`		
`2147483647`	`<`	`2147483648U`		
`2147483647`	`<`	`(int)2147483648U`		

Note: T_{min} = -2,147,483,648 T_{max} = 2,147,483,647

T_{min} must be written as `-2147483647-1` (see pg. 77 of CSAPP for details)