



# Digital Logic

*Gateway to computer science*

transistors, gates, circuits, Boolean algebra

**Software**

**Program, Application**

**Programming Language**

**Compiler/Interpreter**

**Operating System**

**Instruction Set Architecture**

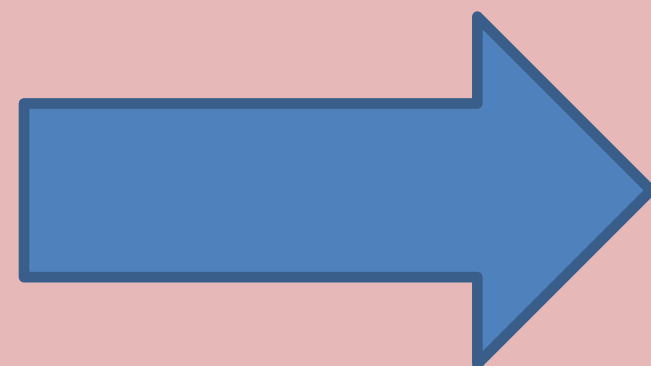
**Hardware**

**Microarchitecture**

**Digital Logic**

**Devices (transistors, etc.)**

**Solid-State Physics**

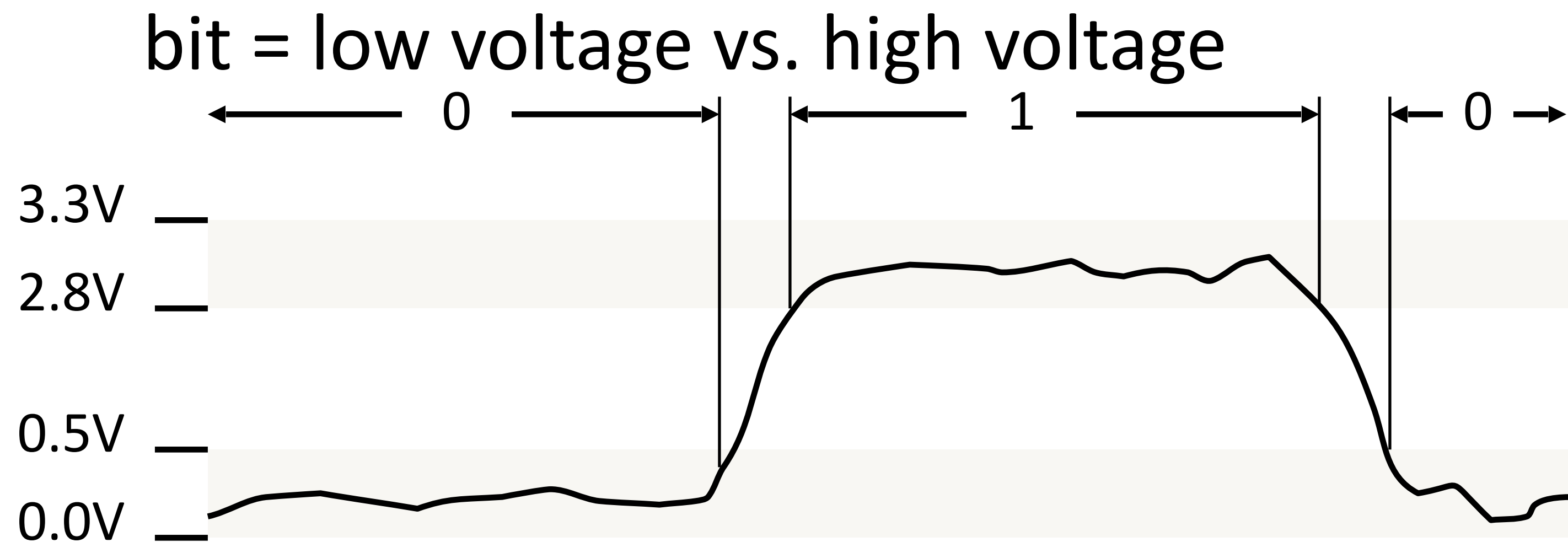


# Digital data/computation = Boolean

Boolean value (*bit*): 0 or 1

Boolean functions (AND, OR, NOT, ...)

Electronically:

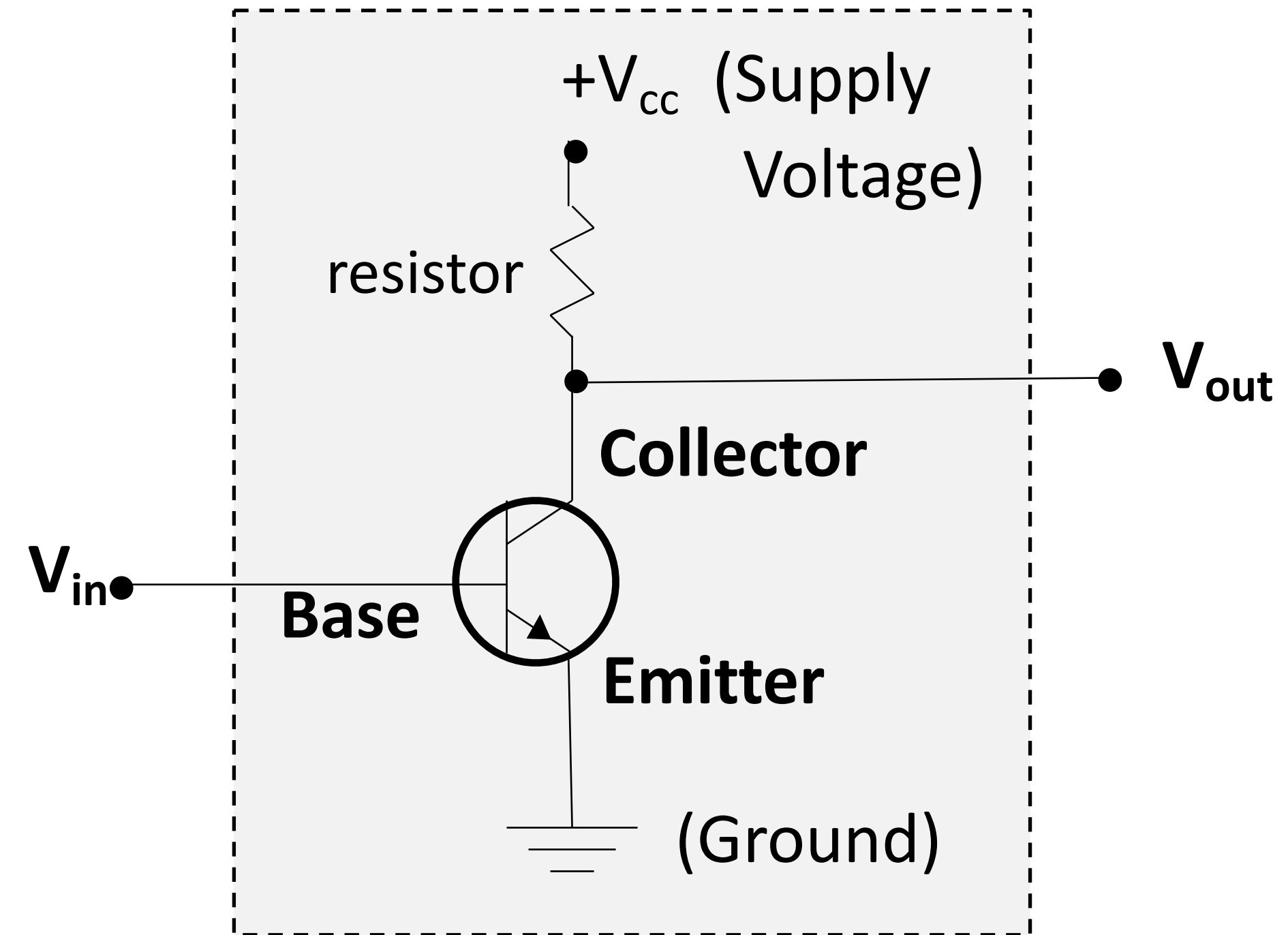


Boolean functions = logic gates, built from transistors

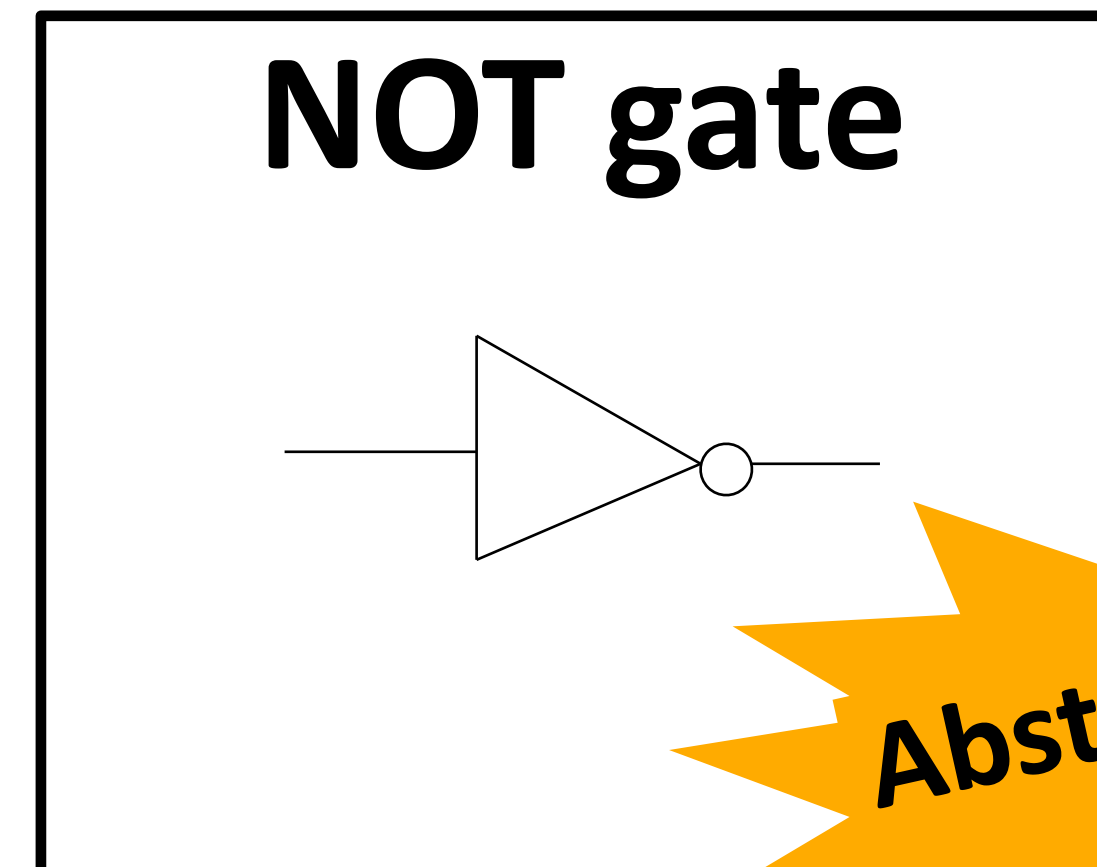
# Transistors

If **Base voltage is high**:  
Current may flow freely  
from *Collector* to *Emitter*.

If **Base voltage is low**:  
Current may not flow  
from *Collector* to *Emitter*.



Truth table							
$V_{in}$	$V_{out}$	=	in	out	=	in	out
low	high	=	0	1	=	F	T
high	low		1	0		T	F



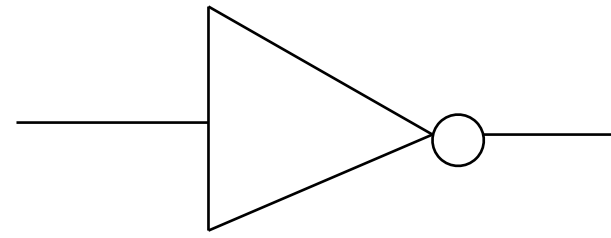
# Digital Logic Gates

Abstraction!

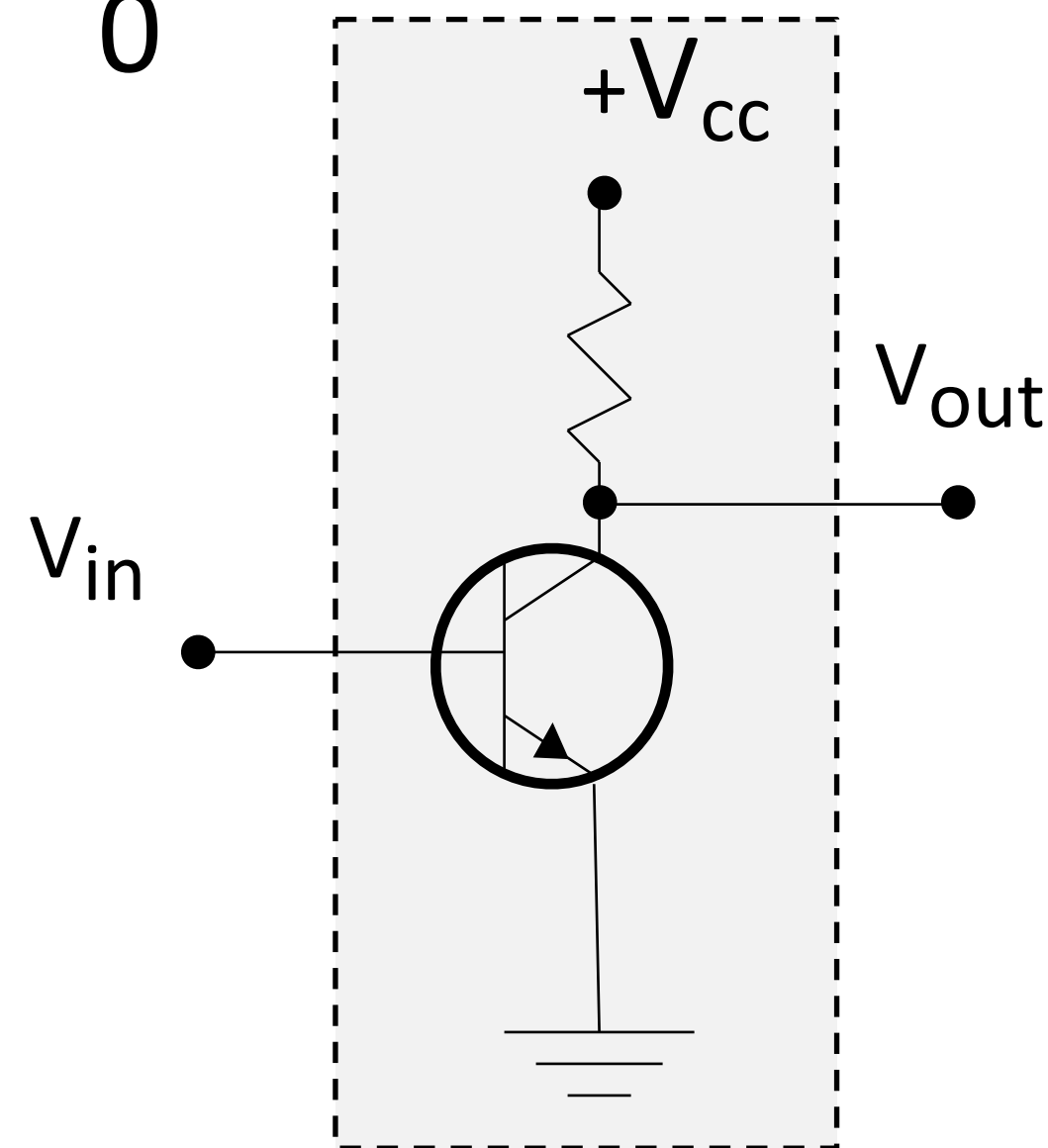
ex

Tiny electronic devices that compute basic Boolean functions.

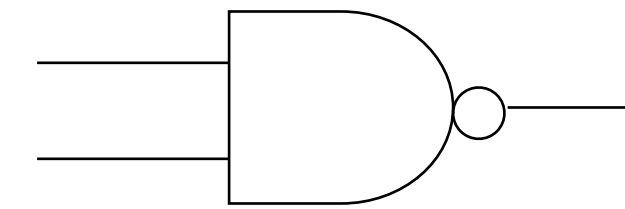
## NOT



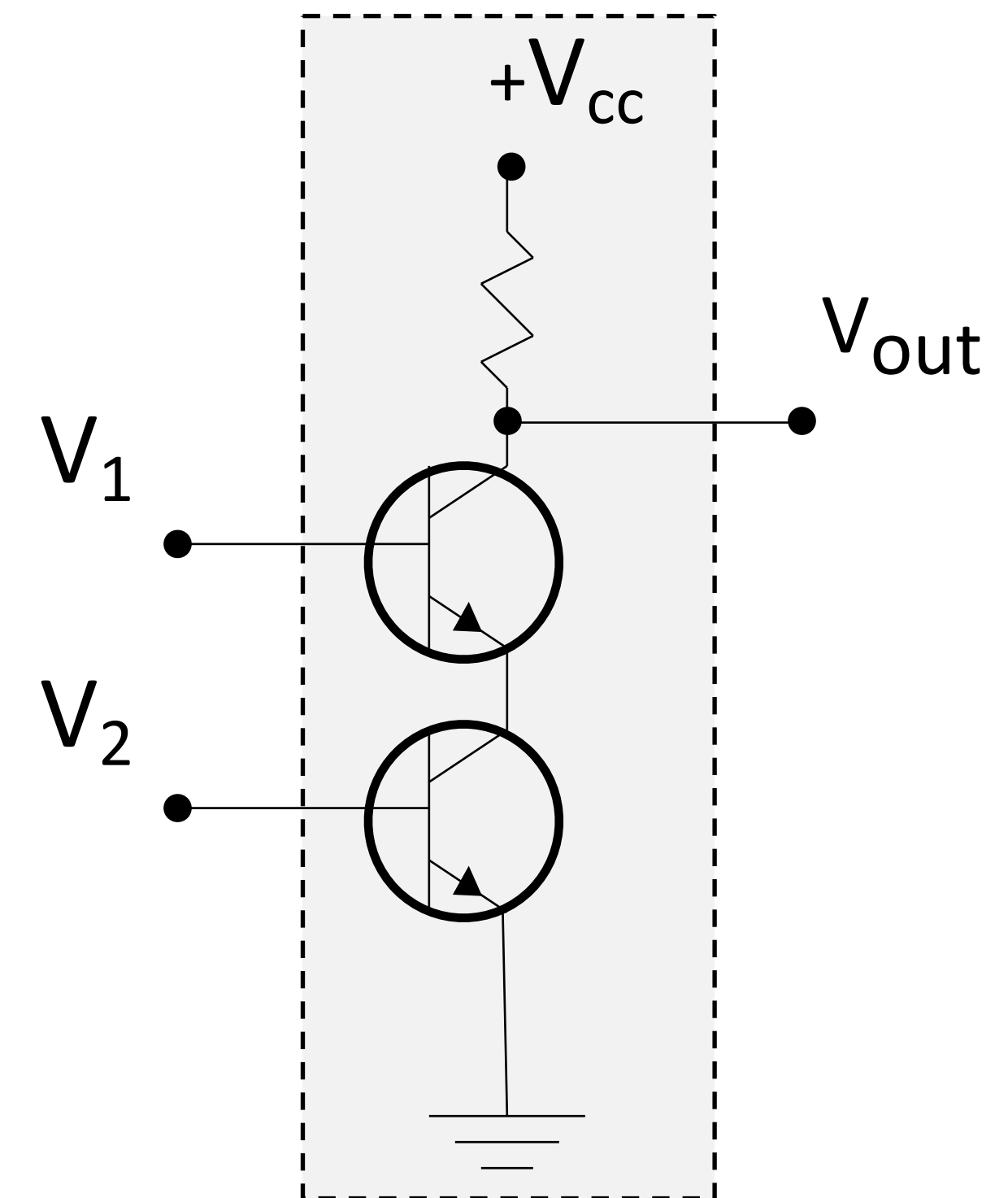
$V_{in}$	$V_{out}$
0	1
1	0



## NAND

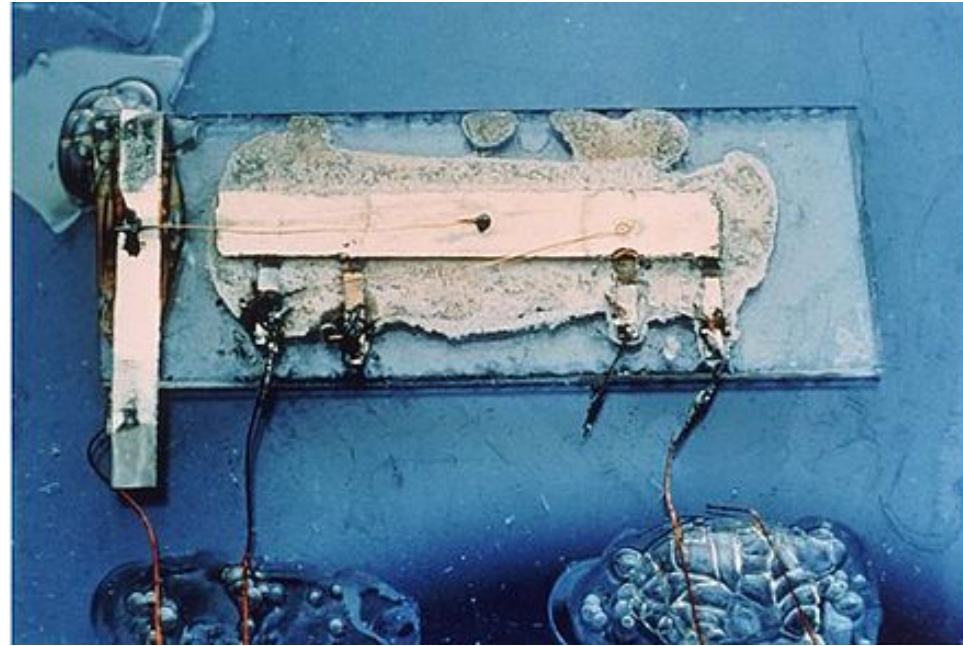


		$V_2$	
		0	1
$V_1$	0		
	1		

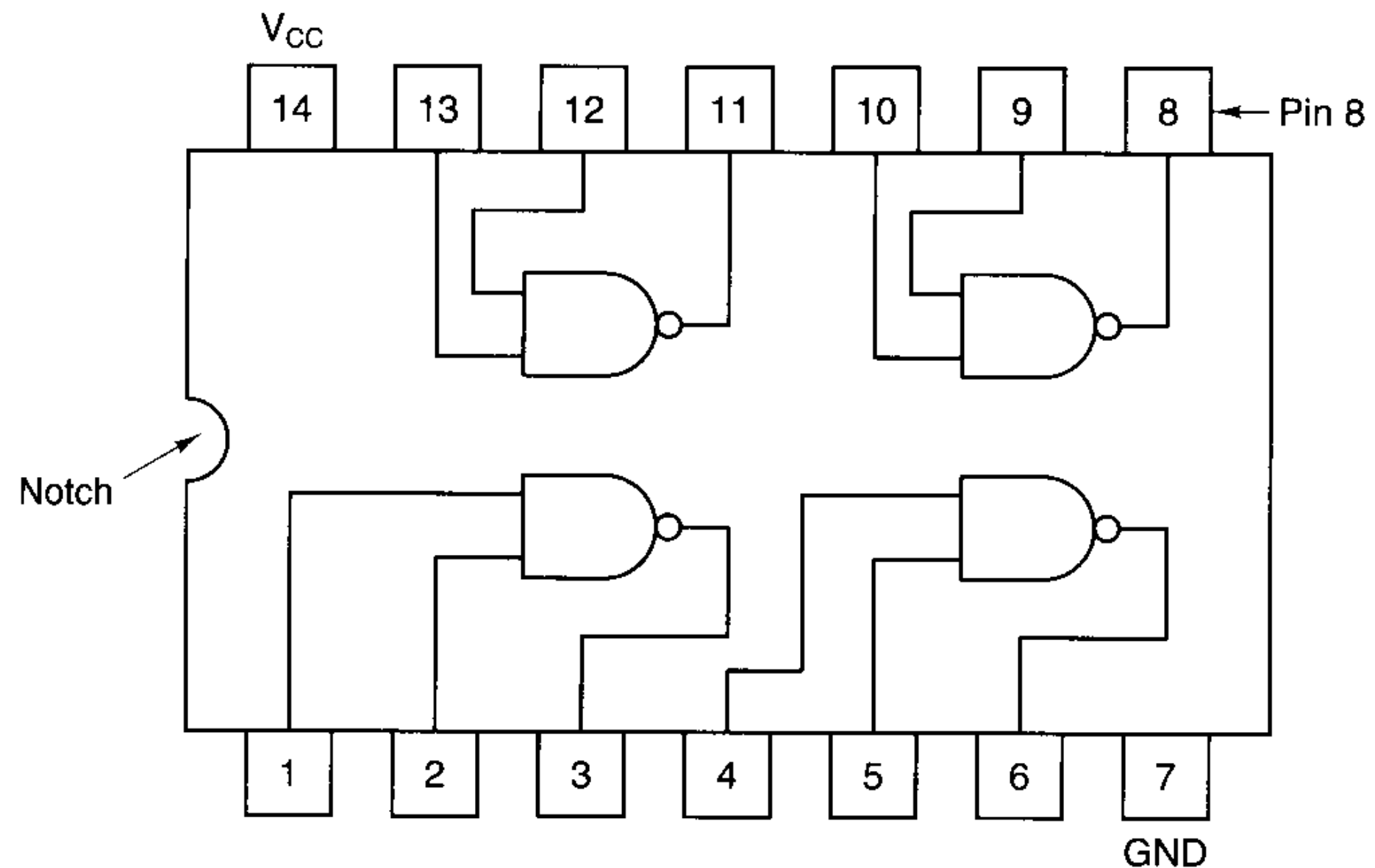


# Integrated Circuits (1950s - )

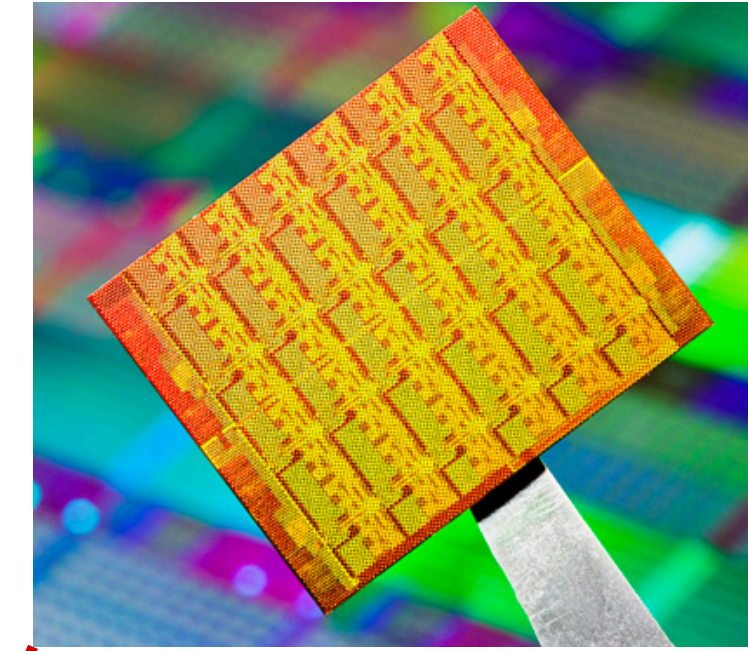
## Early (first?) transistor



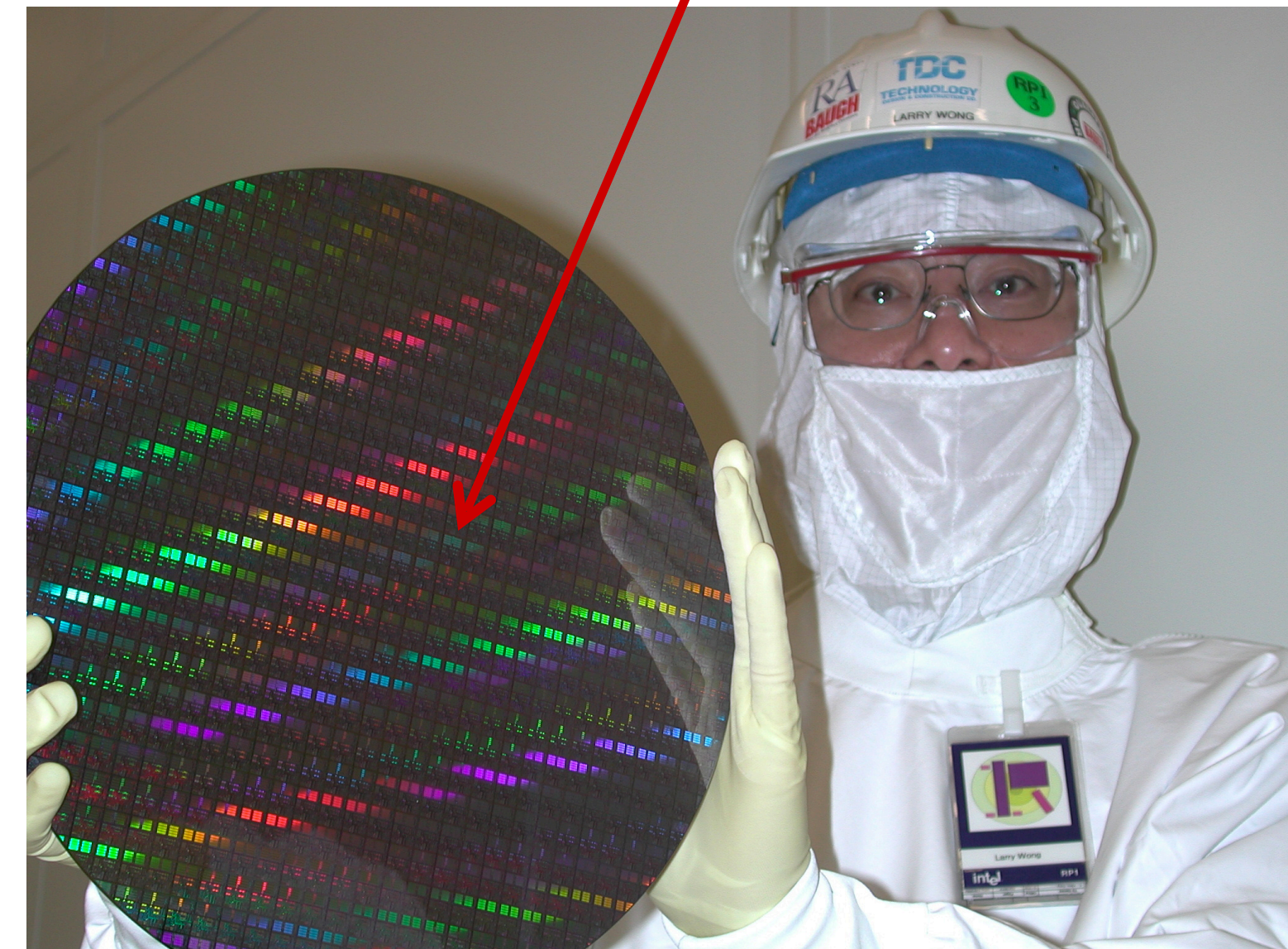
## Small integrated circuit



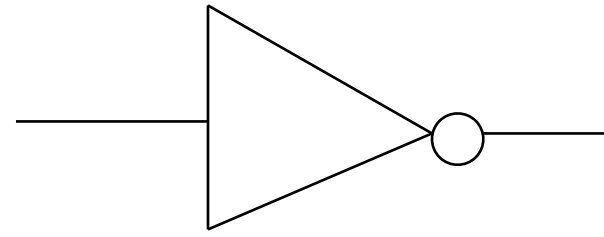
## Chip



## Wafer

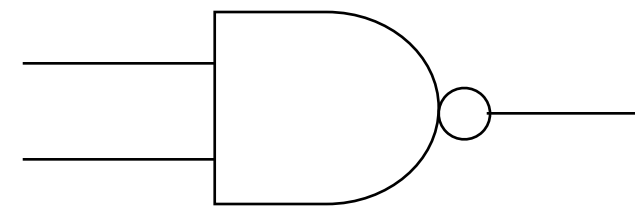


# Five basic gates: define with truth tables



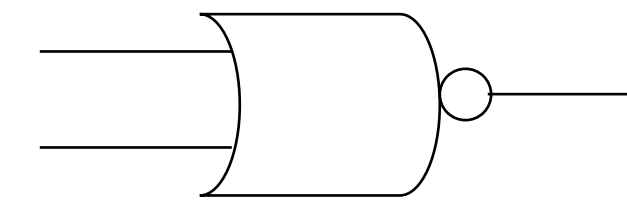
**NOT**

0	1
1	0



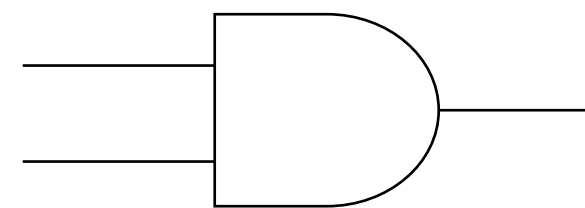
**NAND**

	0	1
0	1	1
1	1	0



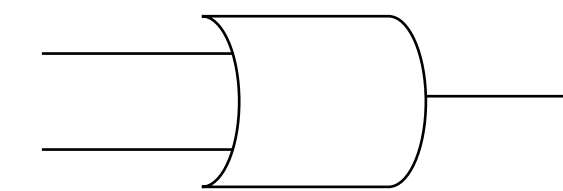
**NOR**

	0	1
0		
1		



**AND**

	0	1
0		
1		

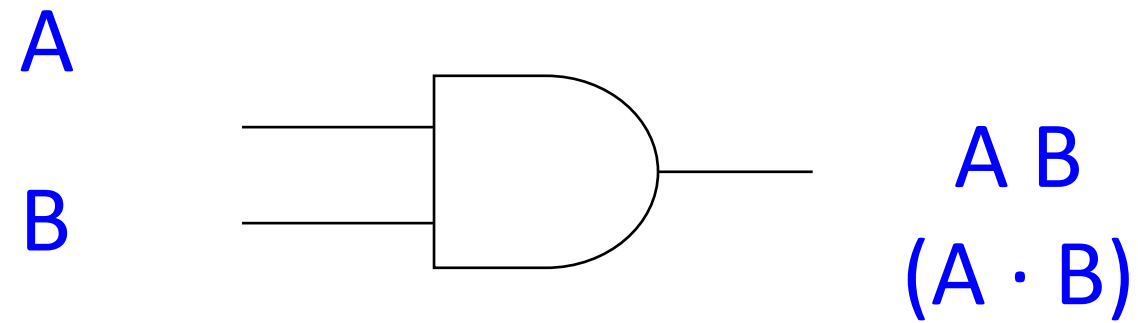


**OR**

	0	1
0		
1		

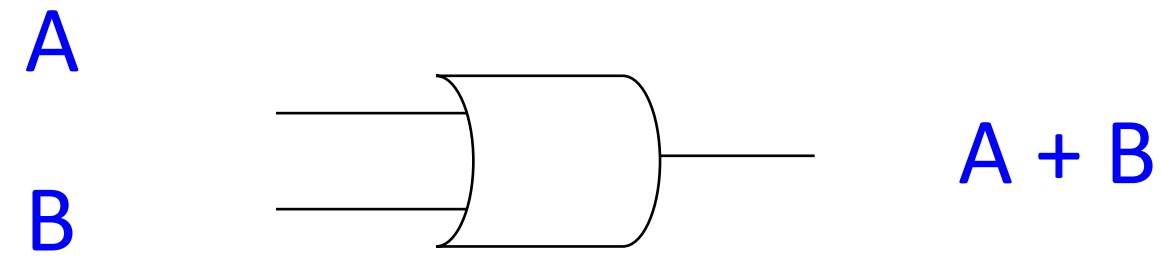
# Boolean Algebra

for combinational logic



AND = Boolean product

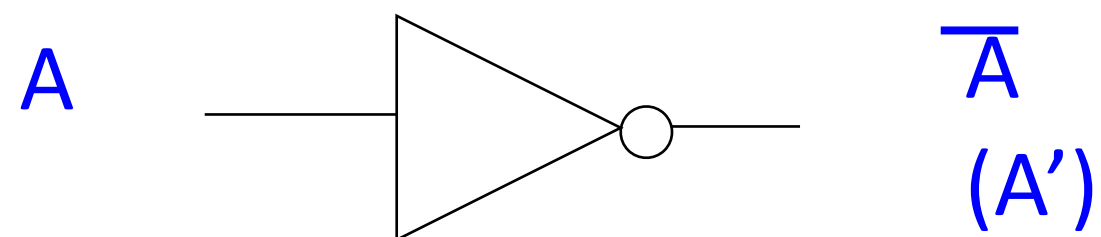
$\cdot$	<b>0</b>	<b>1</b>
<b>0</b>	0	0
<b>1</b>	0	1



OR = Boolean sum

$+$	<b>0</b>	<b>1</b>
<b>0</b>	0	1
<b>1</b>	1	1

*inputs* = *variables*  
*wires* = *expressions*  
*gates* = *operators/functions*  
*circuits* = *functions*



NOT = inverse or complement

	<b>0</b>	<b>1</b>
<b>0</b>	1	0
<b>1</b>	0	1



wire = identity

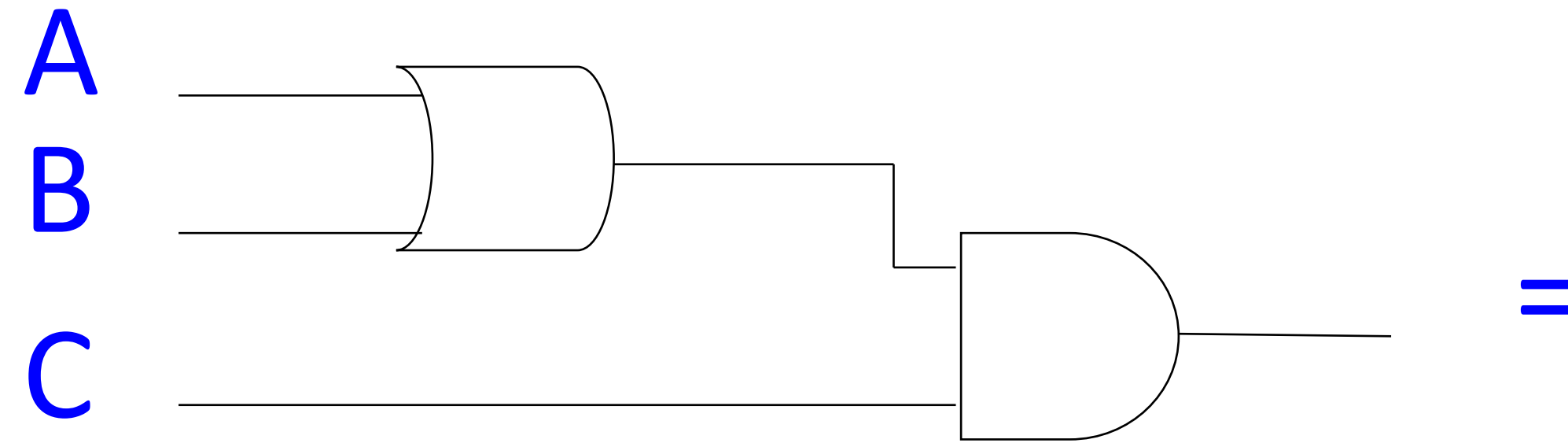
	<b>0</b>	<b>0</b>
<b>0</b>	0	0
<b>1</b>	1	1

A **boolean literal** is a variable or its complement.

E.g.,  $A$ ,  $A'$ ,  $B$ , are literals, but  $A + B$ ,  $AB$ , and  $(AB)'$  are not.



# Circuits

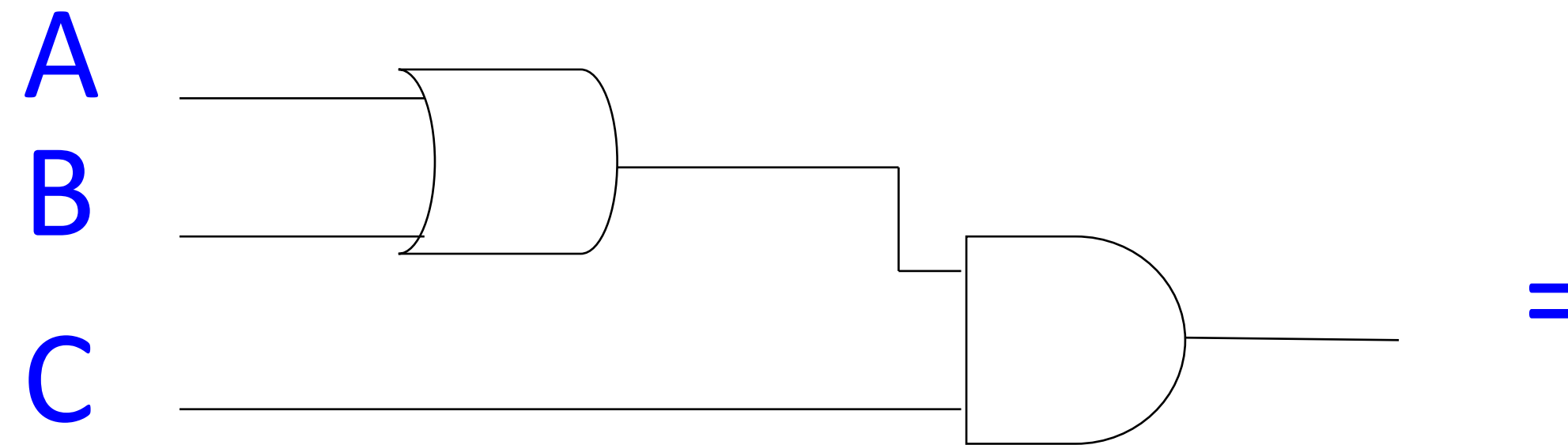


Given input variables, **circuits** specify outputs as functions of inputs using wires & gates.

- Crossed wires touch *only if* there is an explicit dot.
- T intersections copy the value on a wire
- Don't wire together two inputs or two outputs—combine with a gate

# Circuits

ex



What is the truth table for Q in the example circuit?

What is the output if A=1, B=0, C=1?

What is an equivalent Boolean expression?

A	B	C	Q
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

# Translation



Connect gates to implement these functions. Check with truth tables.

Use a direct translation.

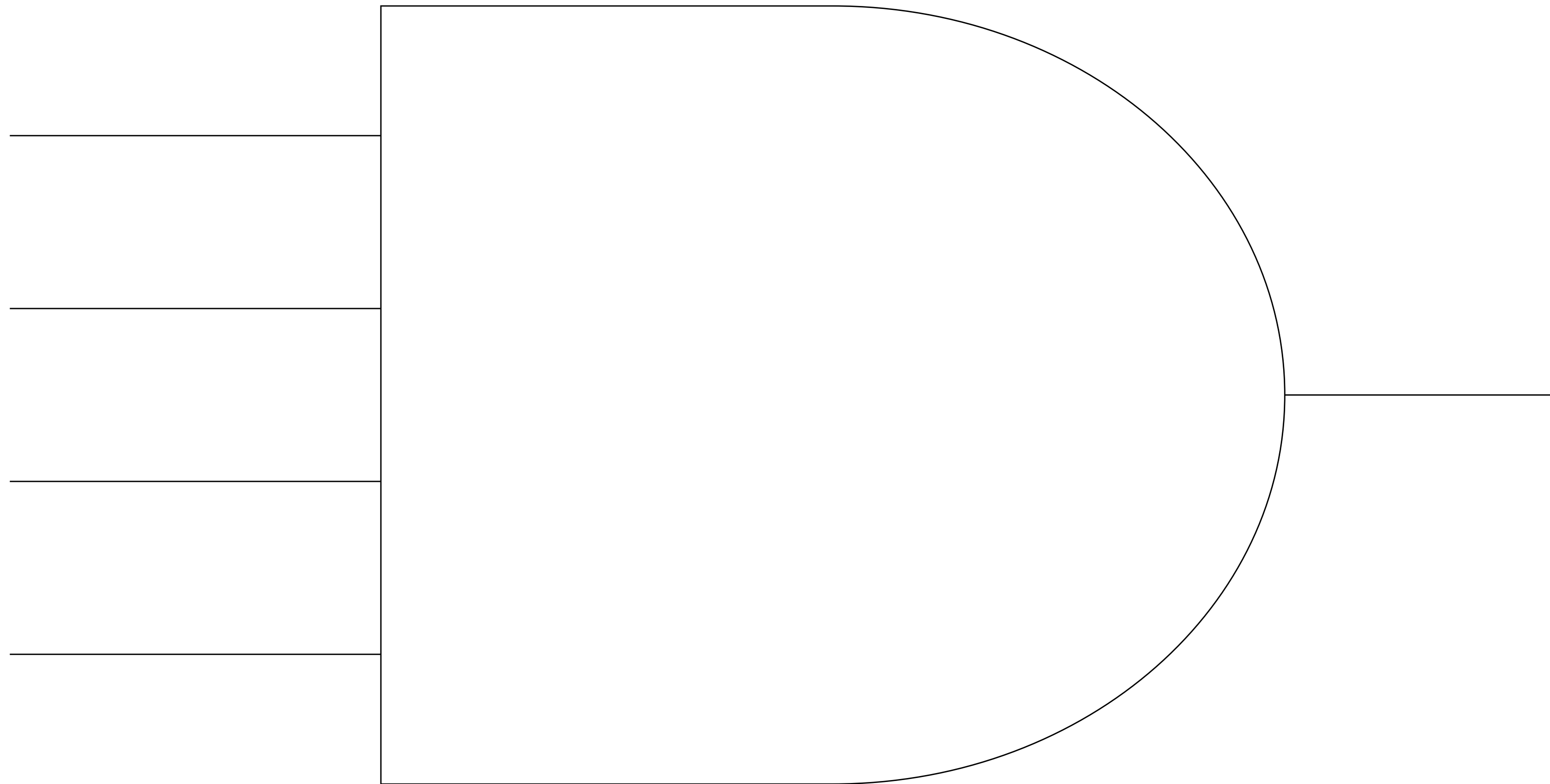
$$F = (A\bar{B} + C)D$$

$$Z = \bar{W} + (X + \bar{W}Y)$$

# Larger gates



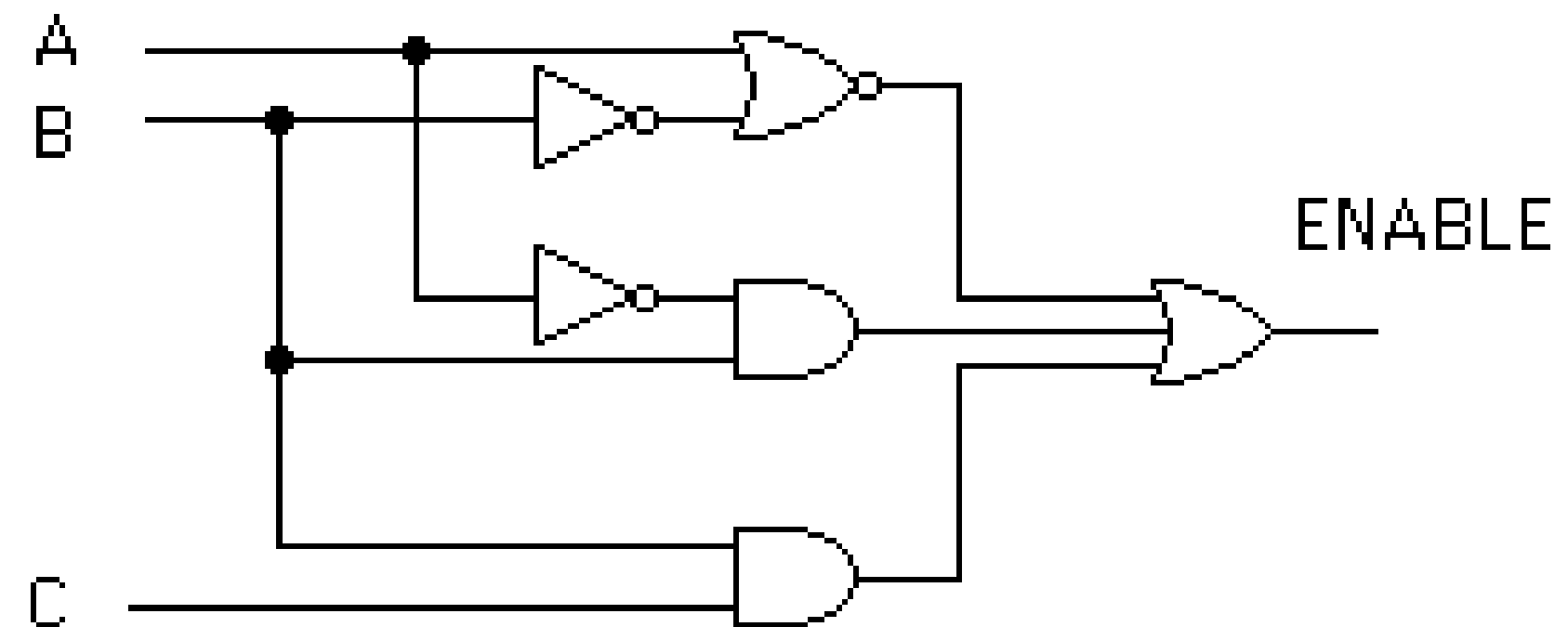
Build a 4-input AND gate using any number of 2-input gates.



# Circuit simplification

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Is there a simpler circuit that performs the same function?



Start with an equivalent Boolean expression, then simplify with algebra.

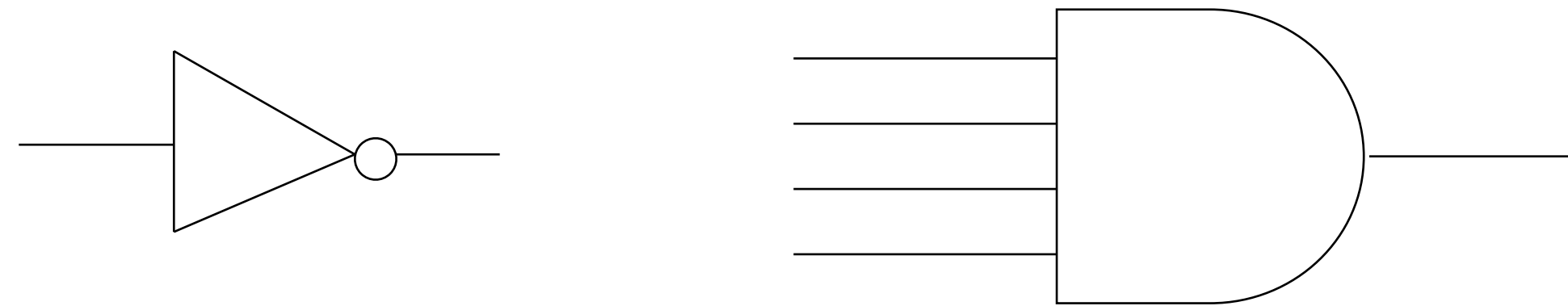
$$F(A, B, C) =$$

Check the answer with a truth table.

# Circuit derivation: *code detectors*



AND gate + NOT gates = code detector, recognizes exactly one input code.



Design a 4-input code detector to output 1 if ABCD = 1001, and 0 otherwise.

A \_\_\_\_\_  
B \_\_\_\_\_  
C \_\_\_\_\_  
D \_\_\_\_\_

Design a 4-input code detector to accept two codes (ABCD=1001, ABCD=1111) and reject all others. (accept = 1, reject = 0)

# Sum-of-products form



logical sum (OR)  
of products (AND)  
of inputs or their negations (NOT)

A **minterm**: a product of literals (variables or their negations) whose value is 1 for that row. Like a **code detector** for that row!

What is the sum-of-products expression for truth table below?

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

How could you draw the circuit for this expression?

How is it related to **code detectors** from the previous slide?

# Voting machines



A majority circuit outputs 1 if and only if a majority of its inputs equal 1.

Design a majority circuit for three inputs. Use a sum of products.

A	B	C	Majority
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

## Triply redundant computers in spacecraft

- [https://en.wikipedia.org/wiki/Triple\\_modular\\_redundancy](https://en.wikipedia.org/wiki/Triple_modular_redundancy)
- Space program hastened Integrated Circuits.



# Boolean Algebra: Simple laws

Boolean algebra laws can be proven by truth tables and used to show equivalences.

For all laws in one place, see the [Boolean Laws Reference Sheet](#)

Name of Law / Theorem	Form	Equivalent/Dual form (interchange AND and OR, and 0 and 1)
<b>Involution</b> (or double negation)	$\overline{\overline{A}} = A$	none
<b>Identity</b>	$0 + A = A$	$1 * A = A$
<b>Inverse</b> (or <b>Complements</b> )	$A\overline{A} = 0$	$A + \overline{A} = 1$
<b>Commutativity</b>	$A + B = B + A$	$AB = BA$
<b>Associativity</b>	$(AB)C = A(BC)$	$(A + B) + C = A + (B + C)$
<b>Idempotent</b>	$A + A = A$	$AA = A$
<b>Null</b> (or <b>Null Element</b> )	$0 * A = 0$ (the <b>Zero Law</b> )	$1 + A = 1$ (the <b>One Law</b> )

# Boolean Algebra: More Complex Laws

<b>Distributive</b>	$A + BC = (A + B)(A + C)$	$A(B + C) = AB + AC$
<b>DeMorgan's</b>	$\overline{A + B + C + \dots} = \overline{A} \overline{B} \overline{C} \dots$	$\overline{\overline{A + B + C + \dots}} = A + B + C + \dots$
<b>Absorption 1 (Covering)</b>	$A + AB = A$	$A(A + B) = A$
<b>Absorption 2</b>	$A + \overline{A}B = A + B$	$A(\overline{A} + B) = AB$
<b>Combining</b>	$AB + \overline{A}\overline{B} = A + B$	$(A + B)(\overline{A} + \overline{B}) = \overline{A} + \overline{B}$
<b>Consensus</b>	$AB + \overline{A}C + BC = AB + \overline{A}C$	$(A + B)(\overline{A} + C)(B + C) = (A + B)(\overline{A} + C)$

You can use truth tables (or other Boolean laws) to convince yourself that these laws hold.

# Boolean Algebra: Proving Laws by Truth Tables



<b>Distributive</b>	$A + BC = (A + B)(A + C)$	$A(B + C) = AB + AC$
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Right column (distributing multiplication over addition) true in the algebra of numbers; left column (distributing addition over multiplication) is **not** true in the algebra of numbers but **is** true for Boolean algebra!

Complete the truth tables below to show that both distributive laws hold.

A	B	C	A+BC
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

A	B	C	(A+B)(A+C)
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

A	B	C	A(B+C)
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

A	B	C	AB + AC
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

# Boolean Algebra: Proving Laws by Algebra

<b>Absorption 1 (Covering)</b>	$A + AB = A$	$A(A + B) = A$
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Each step has a **redex** (subexpression which is changed).

Redex and applied law in the same color here.

*Step-by-step derivation*

*explicit* *Commutativity*

$$\begin{aligned}
 & A + A*B \\
 &= \mathbf{1*A} + A*B \quad [\mathbf{Identity (*)}] \\
 &= \mathbf{A*1} + \mathbf{A*B} \quad [\mathbf{Commutativity (*)}] \\
 &= A*(\mathbf{1 + B}) \quad [\mathbf{Distributivity (*/+)}] \\
 &= \mathbf{A*1} \quad [\mathbf{One law}] \\
 &= \mathbf{1*A} \quad [\mathbf{Commutativity (*)}] \\
 &= A \quad [\mathbf{Identity}]
 \end{aligned}$$

*implicit* *Commutativity*

$$\begin{aligned}
 & A + A*B \\
 &= \mathbf{A*1} + \mathbf{A*B} \quad [\mathbf{Identity}] \\
 &= A*(\mathbf{1 + B}) \quad [\mathbf{Distributivity}] \\
 &= \mathbf{A*1} \quad [\mathbf{One law}] \\
 &= A \quad [\mathbf{Identity}]
 \end{aligned}$$

*Dual step-by-step derivation*

*(Swap \*  $\leftrightarrow$  +, 0  $\leftrightarrow$  1)*

$$\begin{aligned}
 & A*(A + B) \\
 &= (\mathbf{0 + A})* (A + B) \quad [\mathbf{Identity (+)}] \\
 &= (\mathbf{A + 0})* (A + B) \quad [\mathbf{Commutativity (+)}] \\
 &= A + \mathbf{0*B} \quad [\mathbf{Distributivity (+/*)}] \\
 &= \mathbf{A + 0} \quad [\mathbf{Zero law}] \\
 &= \mathbf{0 + A} \quad [\mathbf{Commutativity (+)}] \\
 &= A \quad [\mathbf{Identity}]
 \end{aligned}$$

$$\begin{aligned}
 & A*(A + B) \\
 &= (\mathbf{A + 0})* (A + B) \quad [\mathbf{Identity (+)}] \\
 &= A + \mathbf{0*B} \quad [\mathbf{Distributivity (+/*)}] \\
 &= \mathbf{A + 0} \quad [\mathbf{Zero law}] \\
 &= A \quad [\mathbf{Identity}]
 \end{aligned}$$

# Boolean Algebra: Proving Laws by Algebra

ex

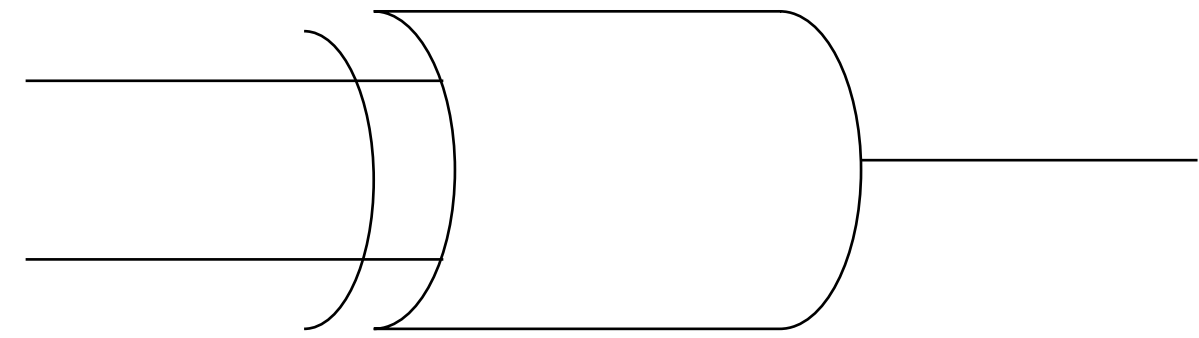
Absorption 2	$A + \bar{A}B = A + B$	$A(\bar{A} + B) = AB$
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# Boolean Algebra: Proving Laws by Algebra

ex

<b>Combining</b>	$AB + A\bar{B} = A$	$(A + B)(A + \bar{B}) = A$
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# XOR: Exclusive OR



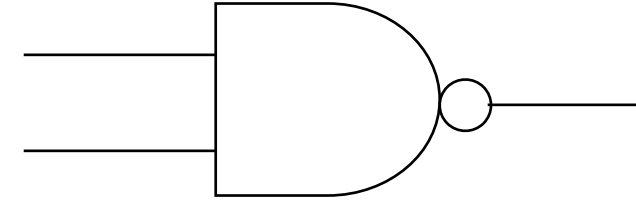
Output = 1 if exactly one input = 1.

Truth table:

Build from earlier gates:

Often used as a one-bit comparator.

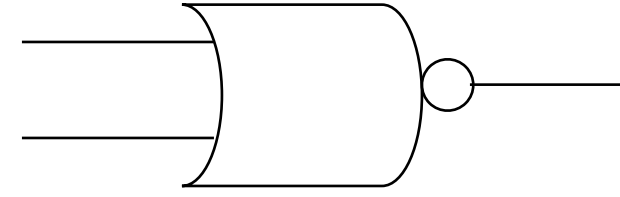
# NAND is *universal*



All Boolean functions can be implemented using only NANDs.  
Build NOT, AND, OR, NOR, using only NAND gates.



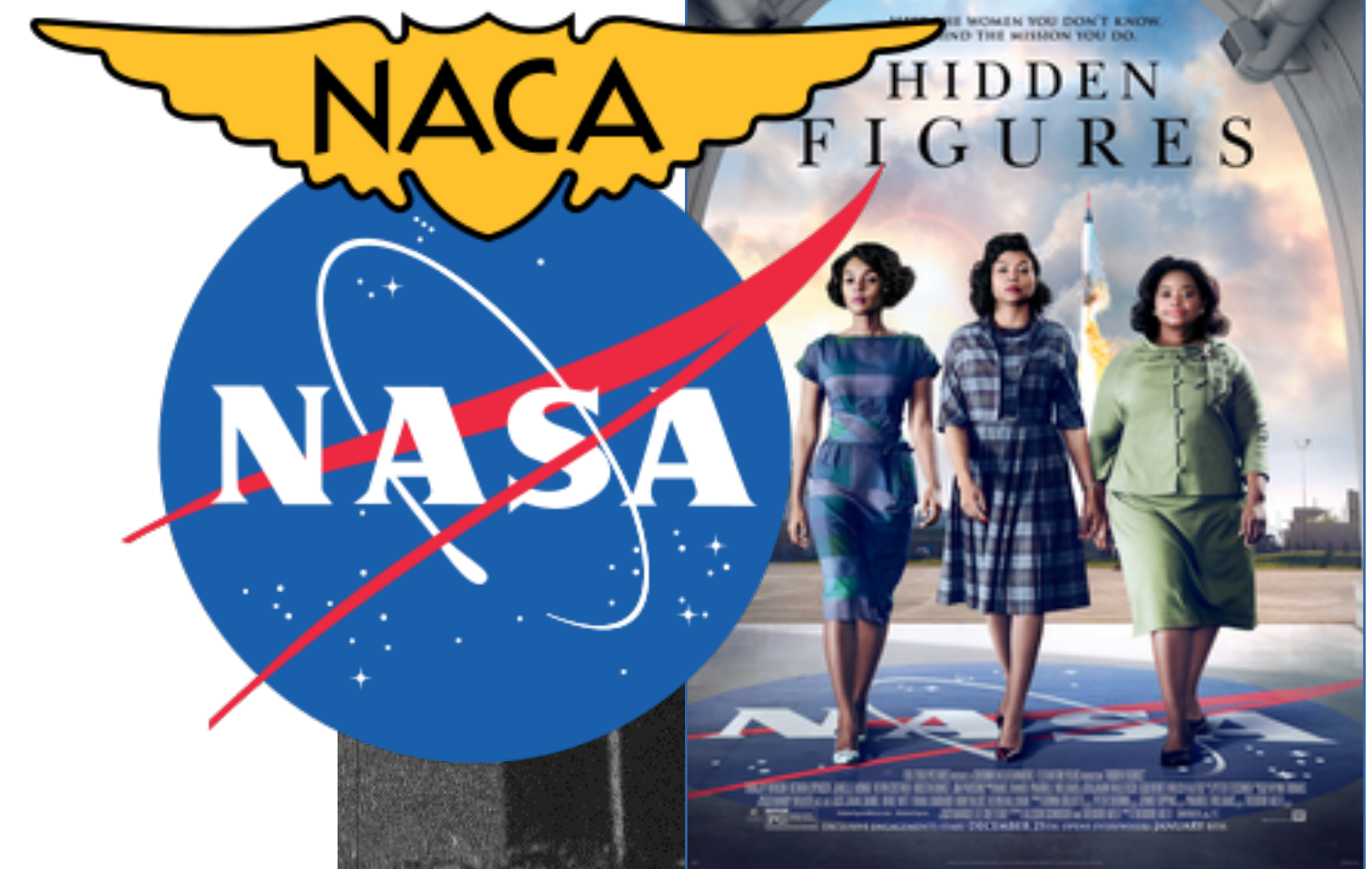
# NOR is also *universal*



All Boolean functions can also be implemented using only NORs.  
Build NAND using only NOR gates; then since NAND is universal,  
NOR must be too! (Why?)

# Computers

- Manual calculations
- powered all early US **space** missions.
- Facilitated transition to digital computers.



**Mary Jackson**



**Katherine Johnson**

- Supported Mercury, Apollo, Space Shuttle, ...

**Dorothy Vaughn**

- First black supervisor within NACA
- Early self-taught FORTRAN programmer for NASA move to digital computers.



# Early pioneers in reliable computing



## Katherine Johnson

- Calculated first US human space flight trajectories
- Mercury, Apollo 11, Space Shuttle, ...
- Reputation for accuracy in manual calculations, verified early code
- Called to verify results of code for launch calculations for first US human in orbit
- Backup calculations helped save Apollo 13
- Presidential Medal of Freedom 2015

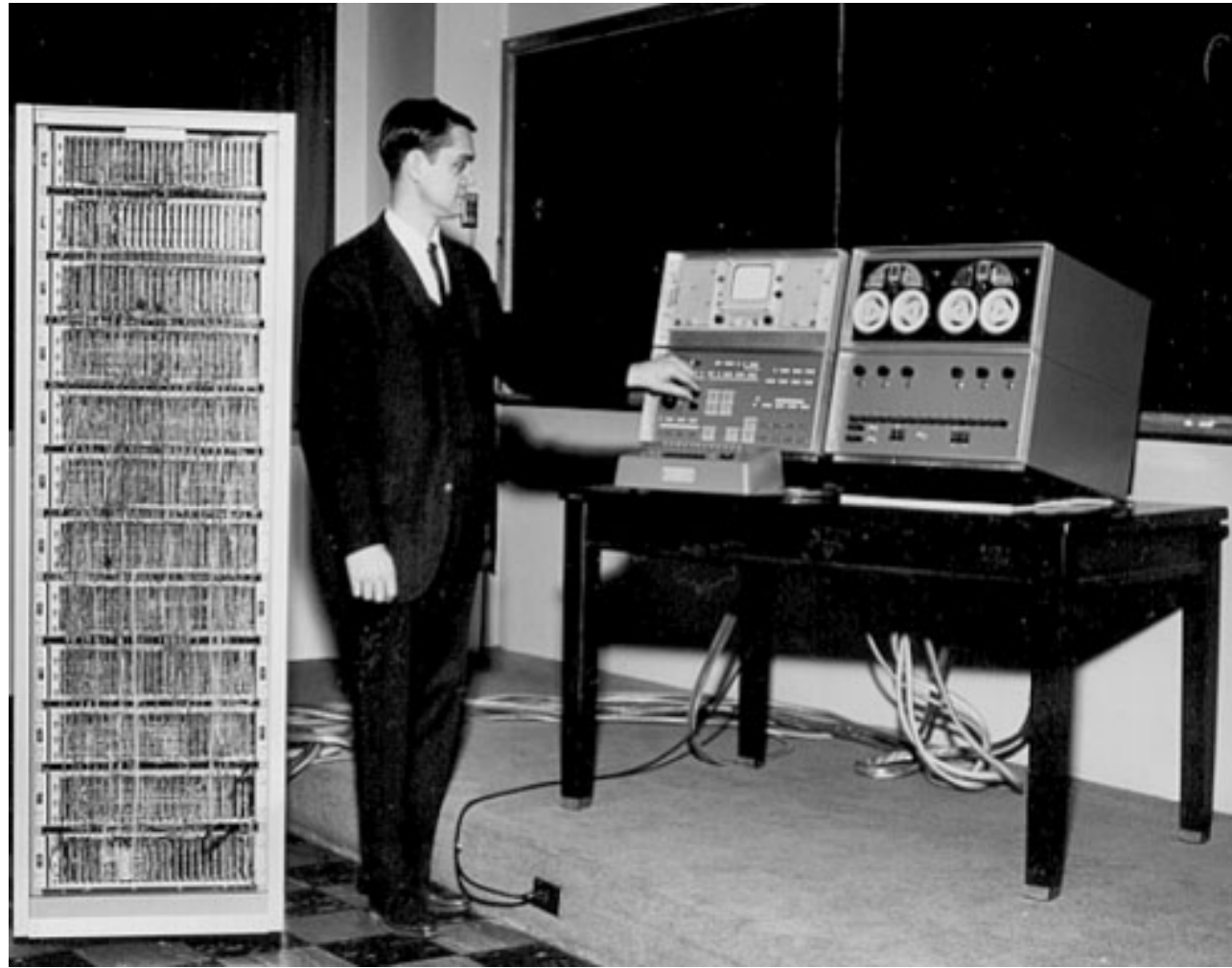
## Margaret Hamilton

- Led software team for Apollo 11 Guidance Computer, averted mission abort on first moon landing.
- Coined “software engineering”, developed techniques for correctness and reliability.
- Presidential Medal of Freedom 2016

## Apollo 11 code print-out



# Wellesley Connection: Mary Allen Wilkes '59



Created LAP operating system at MIT Lincoln Labs for Wesley A. Clark's LINC computer, widely regarded as the first personal computer (designed for interactive use in bio labs). Work done 1961—1965.



Created first interactive keyboard-based text editor on 256 character display. LINC had only 2K 12-bit words; (parts of) editor code fit in 1K section; document in other 1K.

In 1965, she developed LAP6 with LINC in Baltimore living room. First home PC user!



Early versions of LAP developed using LINC simulator on MIT TX2 compute, famous for GUI/PL work done by Ivan and Bert Sutherland at MIT.



Later earned Harvard law degree and headed Economic Crime and Consumer Protection Division in Middlesex (MA) County District Attorney's office.