CS 240 Spring 2020
Foundations of Computer Systems
Ben Wood

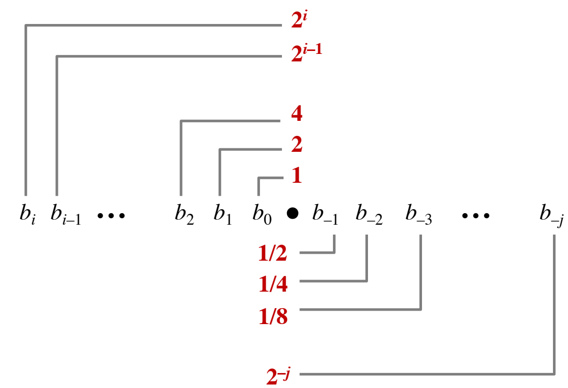
WELLESLEY

Floating Point Representation

Fractional binary numbers
IEEE floating-point standard
Floating-point operations and rounding
Lessons for programmers

Many more details we will skip (it's a 58-page standard...)
See CSAPP 2.4 for more detail.

Fractional Binary Numbers



$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers

Value Representation

- 5 and 3/4
- 2 and 7/8
- 47/64

Observations

- Shift left =
- Shift right =
- Numbers of the form `0.111111`$..._2$ are...?

Limitations:

- Exact representation possible when?

$1/3 = 0.333333..._{10} = 0.$

Fixed-Point Representation

Implied binary point.

$b_7\ b_6\ b_5\ b_4\ b_3$ [.] $b_2\ b_1\ b_0$

$b_7\ b_6\ b_5\ b_4\ b_3\ b_2\ b_1\ b_0$ [.]

range: difference between largest and smallest representable numbers

precision: smallest difference between any two representable numbers

fixed point = fixed range, fixed precision

IEEE **Floating Point** Standard 754

IEEE = Institute of Electrical and Electronics Engineers

Numerical form:

$$V_{10} = (-1)^{s} * M * 2^{E}$$

Sign bit *s* determines whether number is negative or positive

Significand (mantissa) *M* usually a fractional value in range [1.0,2.0)

Exponent *E* weights value by a (-/+) power of two

Analogous to scientific notation

Representation:

MSB **`s`** = sign bit ***s***

`exp` field encodes ***E*** (but is *not equal* to E)

`frac` field encodes ***M*** (but is *not equal* to M)

`s`	`exp`	`frac`

Numerically well-behaved, but hard to make fast in hardware

Precisions

Single precision (`float`): 32 bits

`s`	`exp`	`frac`
1 bit	8 bits	23 bits

Double precision (`double`): 64 bits

`s`	`exp`	`frac`
1 bit	11 bits	52 bits

Finite representation of infinite range...

Three kinds of values

$V = (-1)^{s} * M * 2^{E}$

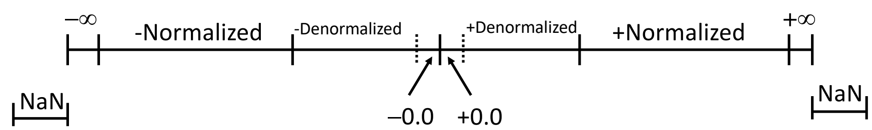
`s`	`exp`	`frac`

1. **Normalized:** *M* = 1.xxxxx...
   As in scientific notation: $0.011 \times 2^5 = 1.1 \times 2^3$
   Representation advantage?

2. **Denormalized**, near zero: *M* = 0.xxxxx..., smallest *E*
   Evenly space near zero.

3. **Special values:**
   0.0: **`s`** = 0 **`exp`** = 00...0 **`frac`** = 00...0
   +inf, -inf: **`exp`** = 11...1 **`frac`** = 00...0
   division by 0.0
   NaN ("Not a Number"): **`exp`** = 11...1 **`frac`** ≠ 00...0
   sqrt(-1), $\infty - \infty$, $\infty * 0$, etc.

Value distribution



Normalized values, with `float` example

$V = (-1)^s * M * 2^E$

s	exp	frac
	k=8	n=23

Value: `float f = 12345.0;`

$12345_{10} = 11000000111001_2$

$= 1.1000000111001_2 \times 2^{13}$ (normalized form)

Significand:

M = `1.1000000111001`$_2$

frac= `10000001110010000000000`$_2$

Exponent: E = **exp** – Bias → **exp** = E + Bias

E = 13

Bias = 127 = $2^7 - 1$ = $\mathbf{2^{k-1} - 1}$ Splits exponents roughly -/+

exp = 140 = `10001100`$_2$

Result:

0	10001100	10000001110010000000000
s	exp	frac

Denormalized Values: near zero

"Near zero": `exp` = `000…0`

Exponent:

E = 1 + `exp` – Bias = 1 - Bias **not:** `exp` – Bias

Significand: leading zero

M = `0.xxx…x`$_2$

frac = xxx…x

Cases:

exp = 000…0, **frac** = 000…0 0.0, -0.0

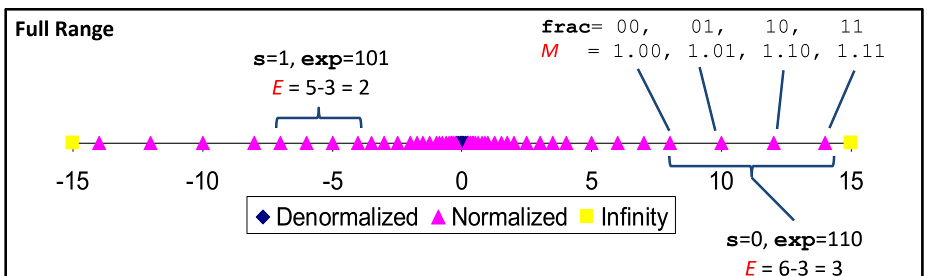
exp = 000…0, **frac** ≠ 000…0

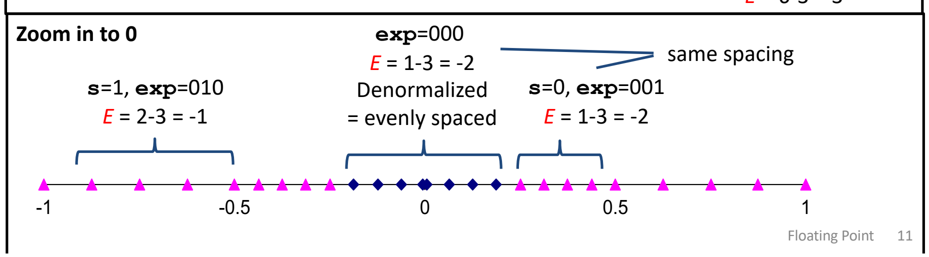
Value distribution example

6-bit IEEE-like format

s	exp	frac
1	3	2

Bias = $2^{3-1} - 1 = 3$





Try to represent 3.14, 6-bit example

6-bit IEEE-like format

Bias = $2^{3-1} - 1 = 3$

s	exp	frac
1	3	2

Value: `3.14;`

3.14 = 11.0010 0011 1101 0111 0000 1010 000…

= 1.1001 0001 1110 1011 1000 0101 0000… $_2 \times 2^1$ (normalized form)

Significand:

M = `1.10010001111010111011100001010000…`$_2$

frac= `10`$_2$

Exponent:

E = 1 Bias = 3 **exp** = 4 = `100`$_2$

Result:

0 100 10 = $1.10_2 \times 2^1 = 3$ next highest?

Floating Point Arithmetic*

$V = (-1)^S * M * 2^E$

s	exp	frac

```
double x = ..., y = ...;
double z = x + y;
```

1. **Compute exact result.**
2. **Fix/Round**, roughly:
   - Adjust *M* to fit in [1.0, 2.0)...
     - If *M* >= 2.0: shift *M* right, increment *E*
     - If *M* < 1.0: shift *M* left by k, decrement *E* by k
   - Overflow to infinity if *E* is too wide for **`exp`**
   - Round* *M* if too wide for **`frac`**.
   - Underflow if nearest representable value is 0.
   - ...

*complicated...

Lessons for programmers

$V = (-1)^S * M * 2^E$

s	exp	frac

`float` ≠ real number ≠ `double`

Rounding breaks associativity and other properties.

```
double a = ..., b = ...;
...
if (a == b) ...

if (abs(a - b) < epsilon) ...
```