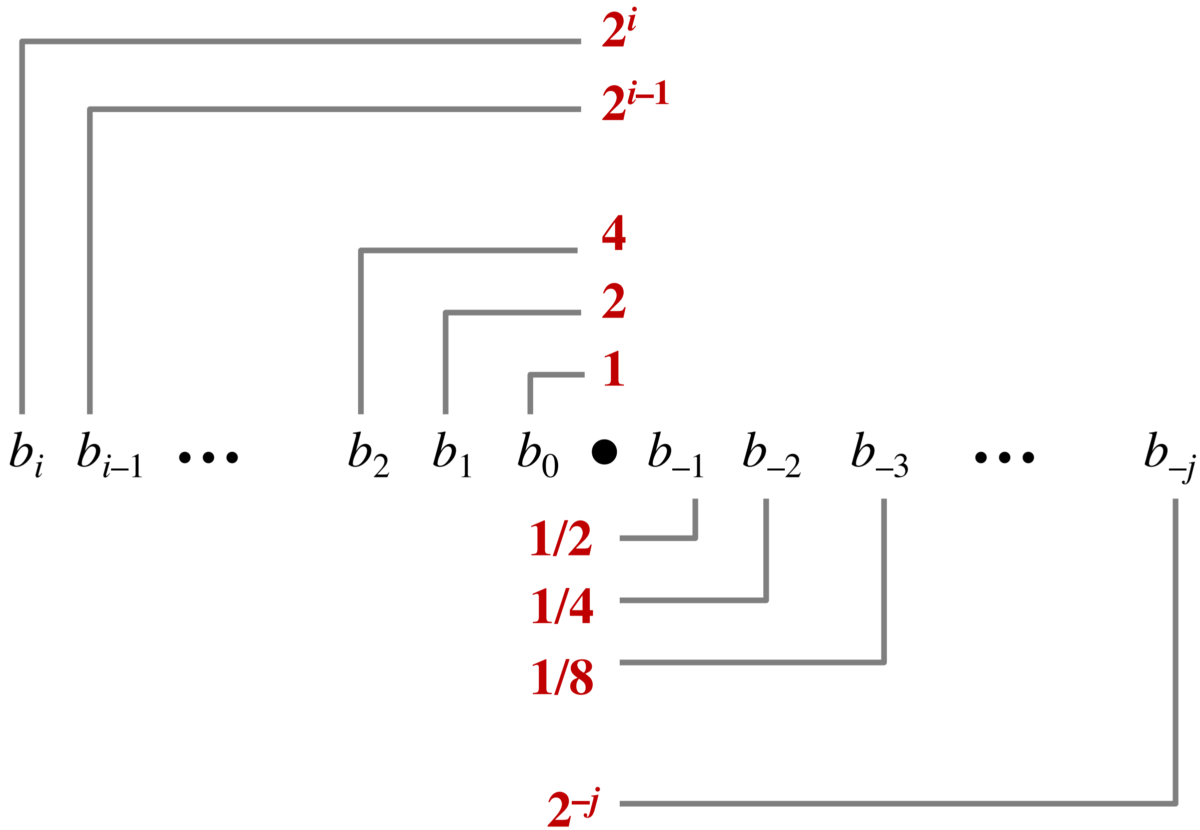
Floating Point Representation

Fractional binary numbers
IEEE floating-point standard
Floating-point operations and rounding
Lessons for programmers

Many more details we will skip (it's a 58-page standard...)
See CSAPP 2.4 for more detail.

Fractional Binary Numbers



$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers

Value | Representation

5 and 3/4

2 and 7/8

47/64

Observations

Shift left =

Shift right =

Numbers of the form `0.111111`$\ldots_2$ are…?

Limitations:

Exact representation possible when?

$1/3 = 0.333333\ldots_{10} = 0.$

Fixed-Point Representation

Implied binary point.

$b_7\ b_6\ b_5\ b_4\ b_3\ \ [.]\ b_2\ b_1\ b_0$

$b_7\ b_6\ b_5\ b_4\ b_3\ b_2\ b_1\ b_0\ [.]$

range: difference between largest and smallest representable numbers

precision: smallest difference between any two representable numbers

fixed point = fixed range, fixed precision

IEEE **Floating Point** Standard 754

IEEE = Institute of Electrical and Electronics Engineers

Numerical form:

$$V_{10} = (-1)^{s} * M * 2^{E}$$

Sign bit ***s*** determines whether number is negative or positive

Significand (mantissa) ***M*** usually a fractional value in range [1.0,2.0)

Exponent ***E*** weights value by a (-/+) power of two

Analogous to scientific notation

Representation:

MSB **`s`** = sign bit ***s***

`exp` field encodes ***E*** (but is *not equal* to E)

`frac` field encodes ***M*** (but is *not equal* to M)

s	exp	frac

Numerically well-behaved, but hard to make fast in hardware

Precisions

Single precision (`float`): 32 bits

s	exp	frac
1 bit	8 bits	23 bits

Double precision (`double`): 64 bits

s	exp	frac
1 bit	11 bits	52 bits

Finite representation of infinite range…

Three kinds of values

$V = (-1)^s * M * 2^E$

s	exp	frac

1. **Normalized:** M = 1.xxxxx…

   As in scientific notation: $0.011 \times 2^5 = 1.1 \times 2^3$

   Representation advantage?

2. **Denormalized**, near zero: M = 0.xxxxx…, smallest E

   Evenly space near zero.

3. **Special values:**

   0.0: **s** = 0 **exp** = 00…0 **frac** = 00…0

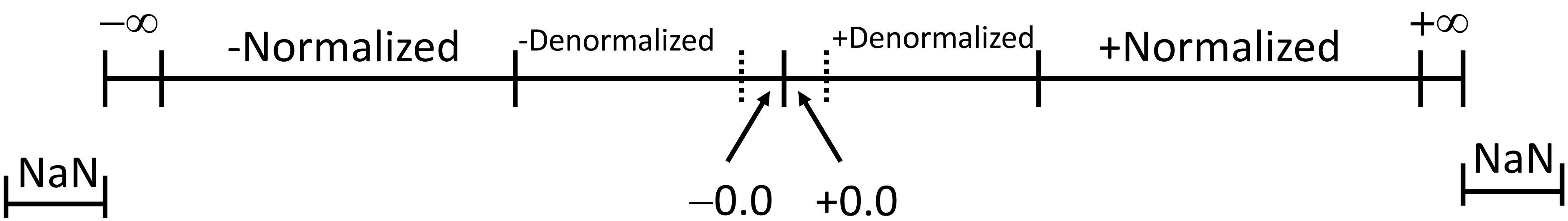
   +inf, -inf: **exp** = 11…1 **frac** = 00…0

   division by 0.0

   NaN ("Not a Number"): **exp** = 11…1 **frac** ≠ 00…0

   sqrt(-1), $\infty - \infty$, $\infty * 0$, etc.

Value distribution



Normalized values, with `float` example

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac
	k=8	n=23

Value: **`float f = 12345.0;`**

$12345_{10} = 11000000111001_2$

$= 1.1000000111001_2 \times 2^{13}$ (normalized form)

Significand:

$M = 1.\underline{1000000111001}_2$

frac $= \underline{1000000111001}0000000000_2$

Exponent: E = **exp** – Bias → **exp** = E + Bias

$E = 13$

Bias $= 127 = 2^7 - 1 = \mathbf{2^{k-1} - 1}$ Splits exponents roughly -/+

exp $= 140 = 10001100_2$

Result:

s	exp	frac
0	10001100	10000001110010000000000

Denormalized Values: near zero

"Near zero": `exp` = `000…0`

Exponent:

E = 1 + `exp` – Bias = 1 - Bias **not:** `exp` – Bias

Significand: leading zero

M = $0.\mathtt{xxx...x}_2$

`frac = xxx…x`

Cases:

`exp` = **`000…0`**, **`frac`** = **`000…0`** 0.0, -0.0

`exp` = **`000…0`**, **`frac`** ≠ **`000…0`**

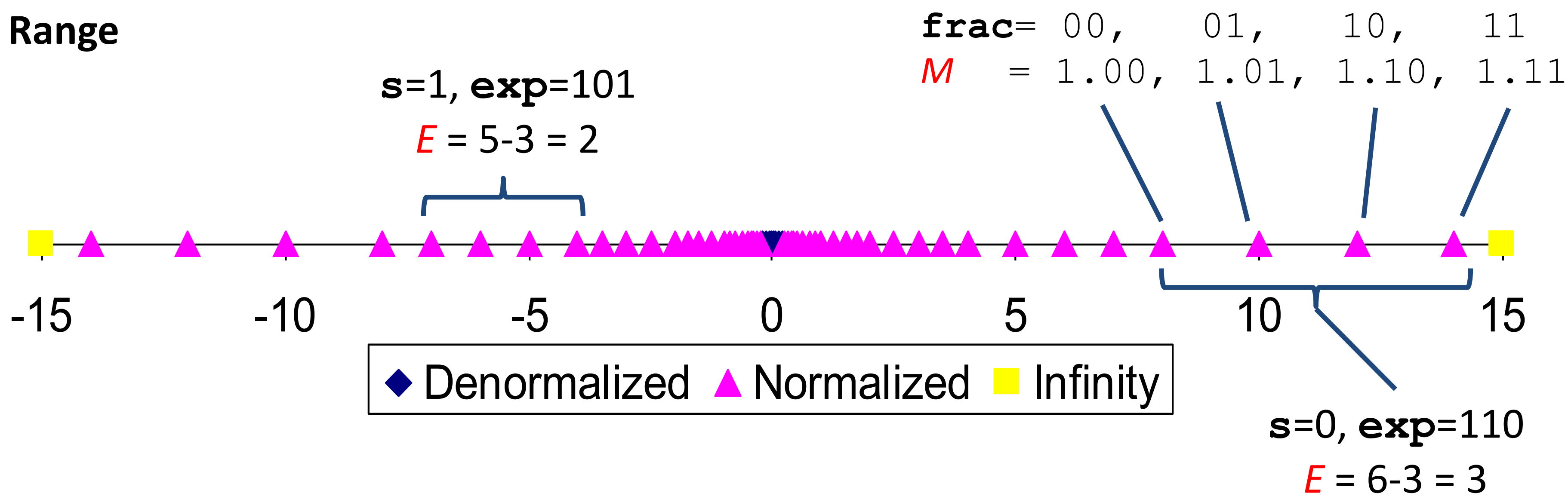
Value distribution example

6-bit IEEE-like format

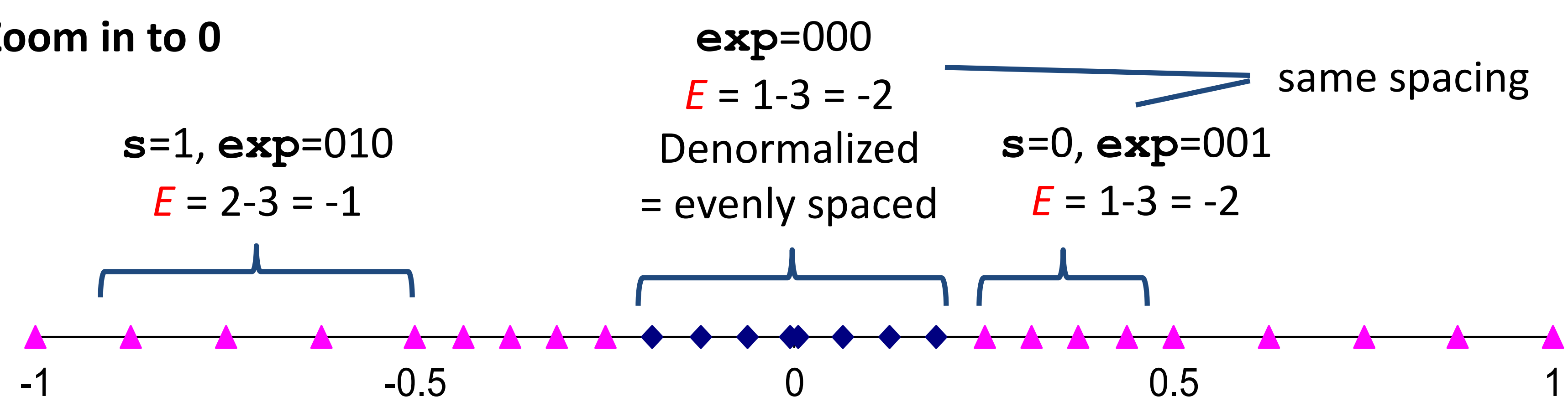
s	exp	frac
1	3	2

Bias = $2^{3-1} - 1 = 3$

Full Range



Zoom in to 0



Try to represent 3.14, 6-bit example

6-bit IEEE-like format

Bias = $2^{3-1} - 1 = 3$

s	exp	frac
1	3	2

Value: **`3.14;`**

$3.14 = 11.0010\ 0011\ 1101\ 0111\ 0000\ 1010\ 000\ldots$

$= 1.1001\ 0001\ 1110\ 1011\ 1000\ 0101\ 0000\ldots_2 \times 2^1$ (normalized form)

Significand:

M = **1.1001000111101011100001010000…** $_2$

frac= $\underline{10}_2$

Exponent:

E = 1 Bias = 3 **exp =** 4 = **100**$_2$

Result:

0 100 10 = $1.10_2 \times 2^1 = 3$ next highest?

Floating Point Arithmetic*

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac

```
double x = ..., y = ...;
double z = x + y;
```

1. **Compute exact result.**
2. **Fix/Round**, roughly:

   Adjust *M* to fit in [1.0, 2.0)…

   - If *M* >= 2.0: shift *M* right, increment *E*
   - If *M* < 1.0: shift *M* left by k, decrement *E* by k

   Overflow to infinity if *E* is too wide for **`exp`**

   Round* *M* if too wide for **`frac`**.

   Underflow if nearest representable value is 0.

   … *complicated…

Lessons for programmers

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac

`float` ≠ real number ≠ `double`

Rounding breaks associativity and other properties.

```
double a = ..., b = ...;
...
if (a == b) ...
```

```
if (abs(a - b) < epsilon) ...
```