CS 251 Part 2:
What's in a Type
Standard ML and Static Types
Topics

• Standard ML basics
• Static type system: types and type-checking rules
ML: Meta-Language for Theorem-Proving

Dana Scott, 1969

Logic of Computable Functions (LCF): for stating theorems about programs

Robin Milner, 1972

Logic for Computable Functions (LCF): automated theorem proving for LCF

Theorem proving is a hard search problem.

ML: Meta-Language for writing programs (tactics) to find proofs of theorems (about other programs)

Proof Tactic: Partial function from formula to proof.

Guides proof search, resulting in one of:

- find and return proof
- never terminate
- report an error
Language Support for Tactics

Static type system
  – guarantee correctness of generated proof

Exception handling
  – deal with tactics that fail (Turing Award)
  – make failure explicit, force programmer to deal with it

First-class/higher-order functions
  – compose other tactics
Defining ML

• Focus on static types.
• New syntax.
• Highly familiar semantics
  – Formal definitions only for the new/different.
  – Some of our simplifications in defining Racket match SML perfectly.
• Move faster since we share some formal experience now.
An ML program is a sequence of bindings.

(* My first ML program *)

val x = 34
val y = 17
val z = (x + y) + (y + 2)
val q = z + 1
val abs_of_z = if z < 0 then 0 – z else z
val abs_of_z_simpler = abs z

(* comment: ML has (* nested comments! *) *)
Bindings, types, and environments

A program is a sequence of bindings.

Bindings build two environments:
- static environment maps variable to type before evaluation
- dynamic environment maps variable to value during evaluation

**Type-check** each binding in order:
- using static environment produced by previous bindings
- and extending it with a binding from variable to type

**Evaluate** each binding in order:
- using dynamic environment produced by previous bindings
- and extending it with a binding from variable to value
SML syntax starter

Bindings
\[ b :::= \text{val } x = e \]
\[ \quad \text{fun } x \ (x : t) => e \]

Types
\[ t :::= \text{bool} \mid \text{int} \mid \text{real} \mid \text{string} \]
\[ \quad (t) \mid t * t \mid t -> t \mid \ldots \]

Expressions:
\[ e :::= \ldots \]

Identifiers:
\[ x \]

Meta-syntax
Type environments
\[ T :::= . \mid x : t, T \]
Type-checking judgments

**Bindings:**

\[ T \vdash \text{bindings} \% T' \]

Under static environment $T$, binding $b$ type-checks and produces extended static environment $T'$.

**Expressions:**

\[ T \vdash e : t \]

Under static environment $T$, expression $e$ type-checks with type $t$. 
Variable bindings

Syntax:

\[
\text{val } x = e \\
\text{val } x = e; \\
\]

Variable name \(x\) and expression \(e\).

Type checking:

If the expression, \(e\), type-checks with type \(t\) under the current static environment, \(T\), then the binding is well-typed and extends the static environment with typing \(x : t\).

\[
\frac{T \vdash e : t}{T \vdash \text{val } x = e \% x : t, T} \quad \text{[t-val]}
\]

Evaluation (only if it type-checks):

\[
\frac{E \vdash b \downarrow E'}{E \vdash \text{val } x = e \% x \mapsto v, E} \quad \text{[e-val]}
\]

Optional semicolon can improve messages for syntax errors.
Expression type-checking rules

\[ T \vdash e : t \]

Value examples:
\[ T \vdash 34 : \text{int} \quad T \vdash \neg1 : \text{int} \]
\[ T \vdash 3.14159 : \text{real} \]
\[ T \vdash \text{true} : \text{bool} \quad T \vdash \text{false} : \text{bool} \]

Variables:
Under static environment \( T \), variable reference \( x \) type-checks with type \( t \) if the static environment maps \( x \) to \( t \).

\[ T(\mathbf{x}) = t \quad \frac{}{T \vdash x : t} \text{[t-var]} \]
Binary expression type-checking rules

Syntax:

```
<table>
<thead>
<tr>
<th>Syntax</th>
<th>e1 + e2</th>
<th>e1 &lt; e2</th>
<th>e1 = e2</th>
<th>e1 &lt;&gt; e2</th>
</tr>
</thead>
</table>
```

Type checking:

<table>
<thead>
<tr>
<th>Type checking</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T \vdash e : t$</td>
<td></td>
</tr>
<tr>
<td>$T \vdash e_1 : \text{int}$</td>
<td>[t-add]</td>
</tr>
<tr>
<td>$T \vdash e_2 : \text{int}$</td>
<td></td>
</tr>
<tr>
<td>$T \vdash e_1 + e_2 : \text{int}$</td>
<td></td>
</tr>
</tbody>
</table>

| $T \vdash e_1 : \text{int}$ | [t-less]  |
| $T \vdash e_2 : \text{int}$ |          |
| $T \vdash e_1 < e_2 : \text{bool}$ |          |

| $T \vdash e_1 : t$ | [t-eq]  |
| $T \vdash e_2 : t$ |          |
| $T \vdash e_1 = e_2 : \text{bool}$ |          |

| $T \vdash e_1 : t$ | [t-ne]  |
| $T \vdash e_2 : t$ |          |
| $T \vdash e_1 <> e_2 : \text{bool}$ |          |

(One more restriction later)
if expressions

Syntax: \[\text{if } e_1 \text{ then } e_2 \text{ else } e_3\]

Type checking:
\[
\begin{align*}
T & \vdash e_1 : \text{bool} \\
T & \vdash e_2 : t \\
T & \vdash e_3 : t \\
T & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : t
\end{align*}
\]

Evaluation:
\[
\begin{align*}
E & \vdash e_1 \downarrow \text{true} \\
E & \vdash e_2 \downarrow v_2 \\
E & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 \downarrow v_2 \\
E & \vdash e_1 \downarrow \text{false} \\
E & \vdash e_3 \downarrow v_3 \\
E & \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 \downarrow v_3
\end{align*}
\]
ML static types and evaluation

Soundness
A program that type-checks never encounters a dynamic type error when evaluated.

Evaluation Rules
Same as our Racket evaluation rules (for ML syntax) except there is no dynamic type checking.
Function examples

(* Anonymous function expression *)
val double = fn (x : int) => x + x
val four = double (2)

(* Function binding *)
fun pow (x : int, y : int) =
  if y = 0
  then 1
  else x * pow (x,y-1)

fun cube (x : int) =
  pow (x,3)

val sixtyfour = cube (four)
val fortytwo =
  pow (2,2+2) + pow (4,2) + cube (2) + 2
A function that takes \( n \) arguments of types \( t_1 \ldots t_n \) and returns a result of type \( t \).
Anonymous function expressions

Syntax:
\[
\text{fn} \ (x_1 : t_1, \ldots, \ x_n : t_n) \Rightarrow e
\]

Type checking:
If the function body, \( e \), type-checks with type \( t \), under the current static environment, \( T \), extended with the argument types, then the function type-checks with type \((t_1 \ast \ldots \ast t_n) \rightarrow t\) under the current static environment, \( T \).

\[
\frac{x_1 : t_1, \ldots, \ x_n : t_n, \ T \vdash e : t}{T \vdash \text{fn} \ (x_1 : t_1, \ldots, x_n : t_n) \Rightarrow e : (t_1 \ast \ldots \ast t_n) \rightarrow t}\]

SML and Static Types
Function bindings

Syntax:

\[
\text{fun } x_0 \ (x_1 : t_1, \ldots, \ x_n : t_n) = e
\]

Type checking:

\[
T \vdash b \% T'
\]

Otherwise equivalent to

\[
\text{val } x_0 = \text{fn} \ (x_1 : t_1, \ldots, \ x_n : t_n) => e
\]

Evaluation: same as Racket.
Function application

Syntax: \( e_0 (e_1, \ldots, e_n) \)

Type checking: \( \Gamma \vdash e : t \)

\[
\begin{align*}
\Gamma & \vdash e_0 : (t_1 \times \cdots \times t_n) \to t \\
\Gamma & \vdash e_1 : t_1 \\
& \quad \vdash \cdots \\
\Gamma & \vdash e_n : t_n \\
\hline \\
\Gamma & \vdash e_0 (e_1, \ldots, e_n) : t
\end{align*}
\]

(* Example *)

fun pow (x : int, y : int) =
  if y = 0
  then 1
  else x \times pow (x, y-1)
Function application

Syntax: \( e_0 (e_1, \ldots, e_n) \)

Evaluation:

1. Under the current dynamic environment, \( E \), evaluate \( e_0 \) to a function closure value \( \langle E', \text{fn} (x_1, \ldots, x_n) \Rightarrow e \rangle \).
   - No dynamic type-checking: Static type-checking guarantees \( e_0 \)'s result value will be a function closure taking parameters \( x_1, \ldots, x_n \) of types matching those of \( e_1, \ldots, e_n \).

2. Under the current dynamic environment, \( E \), evaluate argument expressions \( e_1, \ldots, e_n \) to values \( v_1, \ldots, v_n \).

3. The result is the result of evaluating the closure body, \( e \), under the closure environment, \( E' \), extended with argument bindings: \( x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \).
Function application

Syntax: \[ e0 \ (e1, \ldots, \ en) \]

Evaluation:

\[
\begin{align*}
E \vdash e & \downarrow v \\
E \vdash e0 & \downarrow \langle E', fn (x_1, \ldots, x_n) \Rightarrow e \rangle \\
E \vdash e1 & \downarrow v_1 \\
& \quad \ldots \\
E \vdash en & \downarrow v_n \\
x_1 & \mapsto v_1, \ldots, x_n \mapsto v_n, E' \vdash e \downarrow v \\
\hline
E \vdash e0 \ (e1, \ldots, en) & \downarrow v
\end{align*}
\]

[SML and Static Types]
Watch out

Odd error messages for function-argument syntax errors

* in type syntax is not arithmetic
  – Example: `int * int -> int`
  – In expressions, * is multiplication: `x * pow(x, y-1)`

Cannot refer to later function bindings
  – Helper functions must come before their uses
  – Special construct for mutual recursion (later)
let expressions

... but

Syntax: \[ \text{let } b \text{ in } e \text{ end} \]

- \( b \) is any binding and \( e \) is any expression

Type checking:

\[
\begin{align*}
T \vdash b : T' \\
T' \vdash e : t
\end{align*}
\]

\[
T \vdash \text{let } b \text{ in } e \text{ end} : t
\]  \hspace{1cm} [t-let]

Evaluation:

\[
\begin{align*}
E \downarrow v \vdash \text{let } b \downarrow E' \\
E' \downarrow v \vdash e \downarrow v
\end{align*}
\]

\[
E \downarrow v \vdash \text{let } b \text{ in } e \text{ end} \downarrow v
\]  \hspace{1cm} [e-let]
let is sugar

```
let val x = e1 in e2 end
```

desugars to:

```
((fn (x) => e2) e1)
```

(Rules \([t-let]\) and \([e-let]\) are not needed.)

Multi-binding let:

```
let b1 b2 ... bn in e end
```

desugars to:

```
let b1 in let b2 in ... let bn in e end ... end end
```