In order to understand a program, it is essential to understand the meaning of every name. This requires being able to reliably answer the following question: given a reference occurrence of a name, which binding occurrence does it refer to?

In many cases, the connection between reference occurrences and binding occurrences is clear from the meaning of the binding constructs. For instance, in the HOFL abstraction

\[
\text{fun} \ (a \ b) \ (\text{bind} \ c \ (\text{+} \ a \ b) \ (\text{div} \ c \ 2))
\]

it is clear that the \(a\) and \(b\) within \((+ a b)\) refer to the parameters of the abstraction and that the \(c\) in \((\text{div} \ c \ 2)\) refers to the variable introduced by the \text{bind} expression.

However, the situation becomes murkier in the presence of functions whose bodies have free variables. Consider the following HOFL program:

\[
\text{hofl} \ (a) \\
\quad (\text{bind} \ \text{add-a} \ (\text{fun} \ (x) \ (+ x a)) \\
\quad \quad (\text{bind} \ a \ (+ a 10) \\
\quad \quad \quad \ (\text{add-a} \ (+ 2 a))))
\]

The \text{add-a} function is defined by the abstraction \((\text{fun} \ (x) \ (+ x a))\), which has a free variable \(a\). The question is: which binding occurrence of \(a\) in the program does this free variable refer to? Does it refer to the program parameter \(a\) or the \(a\) introduced by the \text{bind} expression?

A scoping mechanism determines the binding occurrence in a program associated with a free variable reference within a function body. In languages with block structure\(^1\) and/or higher-order functions, it is common to encounter functions with free variables. Understanding the scoping mechanisms of such languages is a prerequisite to understand the meanings of programs written in these languages.

We will study two scoping mechanisms in the context of the HOFL language: static scoping (Sec. 1) and dynamic scoping (Sec. 2). To simplify the discussion, we will initially consider HOFL programs that do not use the \text{bindrec} construct. Then we will study recursive bindings in more detail in Sec. 3).

1 Static Scoping

1.1 Contour Model

In static scoping, the meaning of every variable reference is determined by the lexical contour boxes introduced in Handout #30 on BINDEX. To determine the binding occurrence of any reference occurrence of a name, find the innermost contour enclosing the reference occurrence that binds the name. This is the desired binding occurrence.

For example, below is the contour diagram associated with the \text{add-a} example. The reference to \(a\) in the expression \((+ x a)\) lies within contour boxes \(C_1\) and \(C_0\). \(C_1\) does not bind \(a\), but \(C_0\) does.

---

\(^1\)A language has block structure if functions can be declared locally within other functions. As we shall see later in this course, a language can have block structure without having first-class functions.
so the a in (+ x a) refers to the a bound by (hofl (a) ... ). Similarly, it can be determined that:

- the a in (+ a 10) refers to the a bound by (hofl (a) ... );
- the a in (* 2 a) refers the a bound by (bind a ... );
- the x in (+ x a) refers to the x bound by (abs (x) ... ).
- the add-a in (add-a (* 2 a)) refers to the add-a bound by (bind add-a ... ).

Static scoping is also known as **lexical scoping** because the meaning of any reference occurrence is apparent from the lexical structure of the program.

As another example of a contour diagram, consider the contours associated with the following program containing a **create-sub** function:

By the rules of static scope:

- the n in (- x n) refers to the n bound by the (fun (n) ... ) of create-sub;
- the n in (- n 1) refers to the n bound by the (fun (n) ... ) of test;
- the n in (+ n 1) refers to the n bound by (hofl (n) ... ).
1.2 Substitution Model

The same substitution model used to explain the evaluation of OCAML, BINDEX, and VALEX can be used to explain the evaluation of statically scoped HOFL expressions that do not contain bindrec. (Handling bindrec is tricky in the substitution model, and will be considered later.)

For example, suppose we run the program containing the add-a function on the input 3. Then the substitution process yields:

\[
\begin{align*}
\text{hofl (a)} \\
&\quad (\text{bind add-a (fun (x) (+ x a))}) \\
&\quad (\text{bind a (+ a 10)}) \\
&\quad (\text{add-a (** 2 a))})
\end{align*}
\]

; Here and below, assume a `'smart'` substitution that performs renaming only when variable capture is possible.

\[
\Rightarrow (\text{bind add-a (fun (x) (+ x 3))}) \\
&\quad (\text{add-a (** 2 a))})
\]

\[
\Rightarrow (\text{bind a 13 ((fun (x) (+ x 3)) (** 2 a))}) \\
\Rightarrow ((\text{fun (x) (+ x 3)) (** 2 13)) \\
\Rightarrow ((\text{fun (x) (+ x 3)) 26) \\
\Rightarrow (+ 26 3) \\
\Rightarrow 29
\]

As a second example, suppose we run the program containing the create-sub function on the input 12. Then the substitution process yields:

\[
\begin{align*}
\text{hofl (n)} \\
&\quad (\text{bind create-sub (fun (n) (fun (x) (- x n))}) \\
&\quad (\text{bindpar ((sub2 (create-sub 2))}) \\
&\quad (\text{sub3 (create-sub 3))}) \\
&\quad (\text{bind test (fun (n) (sub2 (sub3 (- n 1))))}) \\
&\quad (\text{bindpar ((test (sub3 (+ 12 1)))))})
\end{align*}
\]

\[
\Rightarrow (\text{bind create-sub (fun (n) (fun (x) (- x n))}) \\
&\quad (\text{bindpar ((sub2 (create-sub 2))}) \\
&\quad (\text{sub3 (create-sub 3))}) \\
&\quad (\text{bind test (fun (n) (sub2 (sub3 (- n 1))))}) \\
&\quad (\text{bindpar (test (sub3 13))))})
\]

\[
\Rightarrow (\text{bindpar (sub2 (fun (x) (- x 2))}) \\
&\quad (\text{sub3 (fun (x) (- x 3))}) \\
&\quad (\text{bind test (fun (n) (sub2 (sub3 (- n 1))))}) \\
&\quad (\text{test (sub3 13))))}
\]

\[
\Rightarrow (\text{bind test (fun (n) ((fun (x) (- x 2)) (fun (x) (- x 3)) (- n 1)))) \\
&\quad (\text{test (fun (x) (- x 3) 13))})
\]

\[
\Rightarrow (\text{fun (n) ((fun (x) (- x 2)) (fun (x) (- x 3)) (- n 1))) (fun (x) (- x 3) 13)) \\
\Rightarrow (\text{fun (n) ((fun (x) (- x 2)) (fun (x) (- x 3)) (- n 1))) (- 13 3)) \\
\Rightarrow (\text{fun (n) ((fun (x) (- x 2)) (fun (x) (- x 3)) (- n 1))) 10) \\
\Rightarrow (\text{fun (x) (- x 2)) (fun (x) (- x 3)) (- 10 1)) \\
\Rightarrow (\text{fun (x) (- x 2)) (fun (x) (- x 3)) 9)) \\
\Rightarrow (\text{fun (x) (- x 2)) (- 9 3)) \\
\Rightarrow (\text{fun (x) (- x 2)) 6) \\
\Rightarrow (- 6 2) \\
\Rightarrow 4
\]
We can formalize the HOFL substitution model by defining a substitution model evaluator in OCAML. Fig. 1 presents the abstract syntax and values used by the evaluator as well as the definition of substitution. The evaluator itself is presented in Fig. 2. The third component of a Fun value, an environment, is not used in the substitution model but plays a very important role in the environment model. The omitted bindrec case will be explained later.

1.3 Environment Model

We would like to be able to explain static scoping within the environment model of evaluation. In order to explain the structure of environments in this model, it is helpful to draw an environment as a linked chain of environment frames, where each frame has a set of name/value bindings and each frame has a single parent frame. There is a distinguished empty frame that terminates the chain, much as an empty list terminates a linked list. See Fig. 3 for an example. In practice, we will often omit the empty frame, and instead indicate the last frame in a chain as a frame with no parent frame.

Intuitively, name lookup in an environment represented as a chain of frames is performed as follows:

- if the name appears in a binding in the first frame of the chain, a Some option of its associated value is returned;
- if the name does not appear in a binding in the first frame of the chain, the lookup process continues starting at the parent frame of the first frame;
- if the empty frame is reached, a None option is returned, indicated that the name was not found.

Most evaluation rules of the environment model are independent of the scoping mechanism. Such rules are shown in Fig. 4.

It turns out that any scoping mechanism is determined by how the following two questions are answered within the environment model:

1. What is the result of evaluating an abstraction in an environment?

2. When creating a frame to model the application of a function to arguments, what should the parent frame of the new frame be?

In the case of static scoping, answering these questions yields the following rules:

1. Evaluating an abstraction \texttt{ABS} in an environment \texttt{ENV} returns a closure that pairs together \texttt{ABS} and \texttt{ENV}. The closure “remembers” that \texttt{ENV} is the environment in which the free variables of \texttt{ABS} should be looked up; it is like an “umbilical cord” that connects the abstraction to its place of birth. We shall draw closures as a pair of circles, where the left circle points to the abstraction and the right circle points to the environment:

   \[
   \begin{array}{c}
   \text{ABST} \\
   \text{ENV}
   \end{array}
   \]

2. To apply a closure to arguments, create a new frame that contains the formal parameters of the abstraction of the closure bound to the argument values. The parent of this new frame should be the environment remembered by the closure. That is, the new frame should extend the environment where the closure was born, not (necessarily) the environment in
Figure 1: OCAML data types for the abstract syntax of HOFL.
(* val run : Hofl.pgm -> int list -> valu *)
let rec run (Pgm(fmls, body)) ints =
  let flen = length fmls
  and ilen = length ints
  in
  if flen = ilen then
    eval (substAll (map (fun i -> Lit (Int i)) ints) fmls body)
  else
    raise (EvalError ("Program expected " ^ (string_of_int flen)
                        ^ " arguments but got " ^ (string_of_int ilen)))

(* val eval : Hofl.exp -> valu *)
and eval exp =
  match exp with
  Lit v -> v
| Var name -> raise (EvalError("Unbound variable: " ^ name))
| PrimApp(op, rands) -> (primopFunction op) (map eval rands)
| If(tst, thn, els) ->
    (match eval tst with
      Bool b -> if b then eval thn else eval els
    | v -> raise (EvalError ("Non-boolean test value "
                             ^ (valuToString v)
                             ^ " in if expression"))
    )
| Abs(fml, body) -> Fun(fml, body, Env.empty) (* No env needed in subst. model *)
| App(rator, rand) -> apply (eval rator) (eval rand)
| Bindrec(names, defns, body) -> ... see discussion of bindrec ...

and apply fcn arg =
  match fcn with
  Fun(fml, body, _) -> eval (subst1 (Lit arg) fml body)
    (* Lit converts any argument valu (including lists & functions)
       into a literal for purposes of substitution *)
  | _ -> raise (EvalError ("Non-function rator in application: "
                          ^ (valuToString fcn)))

Figure 2: Substitution model evaluator in HOFL.

Figure 3: An example chain of environment frames.
Program Running Rule

- To run a HOFL program \((\text{hofl} \ (I_1 \ \ldots \ I_n) \ E_{\text{body}})\) on integers \(i_1, \ldots, i_k\), return the result of evaluating \(E_{\text{body}}\) in an environment that binds the formal parameter names \(I_1 \ \ldots \ I_n\) respectively to the integer values \(i_1, \ldots, i_k\).

Expression Evaluation Rules

- To evaluate a literal expression in any environment, return the value of the literal.
- To evaluate a variable reference expression \(I\) expression in environment \(ENV\), return the value of looking up \(I\) in \(ENV\). If \(I\) is not bound in \(ENV\), signal an unbound variable error.
- To evaluate the conditional expression \((\text{if} \ E_1 \ E_2 \ E_3)\) in environment \(ENV\), first evaluate \(E_1\) in \(ENV\) to the value \(V_1\). If \(V_1\) is true, return the result of evaluating \(E_2\) in \(ENV\); if \(V_1\) is false, return the result of evaluating \(E_3\) in \(ENV\); otherwise signal an error that \(V_1\) is not a boolean.
- To evaluate the primitive application \((O \text{rator} \ E_1 \ \ldots \ E_n)\) in environment \(ENV\), first evaluate the operand expressions \(E_1\) through \(E_n\) in \(ENV\) to the values \(V_1\) through \(V_n\). Then return the result of applying the primitive operator \(O_{\text{primop}}\) to the operand values \(V_1\) through \(V_n\). Signal an error if the number or types of the operand values are not appropriate for \(O_{\text{primop}}\).
- To evaluate the function application \((E_{\text{fcn}} \ E_{\text{rand}})\) in environment \(ENV\), first evaluate the expressions \(E_{\text{fcn}}\) and \(E_{\text{rand}}\) in \(ENV\) to the values \(V_{\text{fcn}}\) and \(V_{\text{rand}}\), respectively. If \(V_{\text{fcn}}\) is a function value, return the result of applying \(V_{\text{fcn}}\) to the operand value \(V_{\text{rand}}\). (The details of what it means to apply a function is at the heart of scoping and, as we shall see, differs among scoping mechanisms.) If \(V_{\text{fcn}}\) is not a a function value, signal an error.

Although \text{bind}, \text{bindrec}, and \text{bindseq} can all be “desugared away”, it is convenient to imagine that there are rules for evaluating these constructs directly:

- Evaluating \((\text{bind} \ I_{\text{name}} \ E_{\text{defn}} \ E_{\text{body}})\) in environment \(ENV\) is the result of evaluating \(E_{\text{body}}\) in the environment that results from extending \(ENV\) with a frame containing a single binding between \(I_{\text{name}}\) and the value \(V_{\text{defn}}\) that results from evaluating \(E_{\text{defn}}\) in \(ENV\).
- A \text{bindpar} is evaluated similarly to \text{bind}, except that the new frame contains one binding for each of the name/defn pairs in the \text{bindpar}. As in \text{bind}, all defns of \text{bindpar} are evaluated in the original frame, not the extension.
- A \text{bindseq} expression should be evaluated as if it were a sequence of nested \text{binds}.

Figure 4: Environment model evaluation rules that are independent of the scoping mechanism.

which the closure was called. This creates the right environment for evaluating the body of the abstraction as implied by static scoping: the first frame in the environment contains the bindings for the formal parameters, and the rest of the frames contain the bindings for the free variables.

We will show these rules in the context of using the environment model to explain executions of the two programs from above. First, consider running the \text{add-a} program on the input 3. This evaluates the body of the \text{add-a} program in an environment \(ENV_0\) binding a to 3:

\[
\begin{array}{c}
\text{ENV0} \\
\hline
\text{a} \\
\end{array} \rightarrow 3
\]

To evaluate the \((\text{bind} \ \text{add-a} \ \ldots)\) expression, we first evaluate \((\text{fun} \ (x) \ (+ \ x \ a))\) in \(ENV_0\). According to rule 1 from above, this should yield a closure pairing the abstraction with \(ENV_0\). A new frame \(ENV_2\) should then be created binding \text{add-a} to the closure:
Next the expression (bind a ...) is evaluated in ENV₂. First the definition (+ a 10) is evaluated in ENV₁, yielding 13. Then a new frame ENV₃ is created that binds a to 13:

Finally the function application (add-a (* 2 a)) is evaluated in ENV₃. First, the subexpressions add-a and (* 2 a) must be evaluated in ENV₃; these evaluations yield the add-a closure and 26, respectively. Next, the closure is applied to 26. This creates a new frame ENV₁ binding x to 26; by rule 2 from above, the parent of this frame is ENV₀, the environment of closure; the environment ENV₃ of the function application is simply not involved in this decision.

As the final step, the abstraction body (+ x a) is evaluated in ENV₁. Since x evaluates to 26 in ENV₃ and a evaluates to 3, the final answer is 29.

As a second example of static scoping in the environment model, consider running the create-sub program from the previous section on the input 12. Below is an environment diagram showing all environments created during the evaluation of this program. You should study this diagram carefully and understand why the parent pointer of each environment frame is the way it is. The final answer of the program (which is not shown in the environment model itself) is 4.
In both of the above environment diagrams, the environment names have been chosen to underscore a critical fact that relates the environment diagrams to the contour diagrams. Whenever environment frame $ENV_i$ has a parent pointer to environment frame $ENV_j$ in the environment model, the corresponding contour $C_i$ is nested directly inside of $C_j$ within the contour model. For example, the environment chain $ENV_6 \rightarrow ENV_4 \rightarrow ENV_3 \rightarrow ENV_0$ models the contour nesting $C_6 \rightarrow C_4 \rightarrow C_3 \rightarrow C_0$, and the environment chains $ENV_{2c} \rightarrow ENV_{1a} \rightarrow ENV_0$, $ENV_{2a} \rightarrow ENV_{1b} \rightarrow ENV_0$, and $ENV_{2b} \rightarrow ENV_{1b} \rightarrow ENV_0$ model the contour nesting $C_2 \rightarrow C_1 \rightarrow C_0$.

These correspondences are not coincidental, but by design. Since static scoping is defined by the contour diagrams, the environment model must somehow encode the nesting of contours. The environment component of closures is the mechanism by which this correspondence is achieved. The environment component of a closure is guaranteed to point to an environment $ENV_{birth}$ that models the contour enclosing the abstraction of the closure. When the closure is applied, the newly constructed frame extends $ENV_{birth}$ with a new frame that introduces bindings for the parameters of the abstraction. These are exactly the bindings implied by the contour of the abstraction. Any expression in the body of the abstraction is then evaluated relative to the extended environment.

### 1.4 Interpreter Implementation of Environment Model

Rules 1 and 2 of the previous section are easy to implement in an environment model interpreter. The implementation is shown in Figure 5. Note that it is not necessary to pass `env` as an argument to `funapply`, because static scoping dictates that the call-time environment plays no role in applying the function.
2 Dynamic Scoping

2.1 Environment Model

In dynamic scoping, environments follow the shape of the invocation tree for executing the program. Recall that an invocation tree has one node for every function invocation in the program, and that each node has as its children the nodes for function invocations made directly within its body, ordered from left to right by the time of invocation (earlier invocations to the left). Since bind desugars into a function application, we will assume that the invocation tree contains nodes for bind expressions as well. We will also consider the execution of the top-level program to be a kind of function application, and its corresponding node will be the root of the invocation tree. For example, here is the invocation tree for the add-a program:

```
run (hofl (a) ...)
  bind add-a
    bind a
    invoke add-a
```

As a second example, here is the invocation tree for the create-sub program:

```
run (hofl (n) ...)
  bind create-sub
    invoke create-sub 2
    invoke create-sub 3
    bindpar sub2,sub3
      bind test
        invoke sub3
        invoke test
          invoke sub3
          invoke sub2
```

Figure 5: Essence of static scoping in HOFL.
Note: in some cases (but not the above two), the shape of the invocation tree may depend on the values of the arguments at certain nodes, which in turn depends on the scoping mechanism. So the invocation tree cannot in general be drawn without fleshing out the details of the scoping mechanism.

The key rules for dynamic scoping are as follows:

1. Evaluating an abstraction $ABS$ in an environment $ENV$ just returns $ABS$. In dynamic scoping, there there is no need to pair the abstraction with its environment of creation.

2. To apply a closure to arguments, create a new frame that contains the formal parameters of the abstraction of the closure bound to the argument values. The parent of this new frame should be the environment in which the function application is being evaluated - that is, the environment of the invocation (call), not the environment of creation. This means that the free variables in the abstraction body will be looked up in the environment where the function is called.

Consider the environment model showing the execution of the $add$-$a$ program on the argument 3 in a dynamically scoped version of HOFL. According to the above rules, the following environments are created:

The key differences from the statically scoped evaluation are (1) the name $add$-$a$ is bound to an abstraction, not a closure and (2) the parent frame of $ENV_3$ is $ENV_2$, not $ENV_0$. This means that the evaluation of $(+ x a)$ in $ENV_3$ will yield 39 under dynamic scoping, as compared to 29 under static scoping.

Figure 6 shows an environment diagram showing the environments created when the $create$-$sub$ program is run on the input 12. The top of the figure also includes a copy of the invocation tree to emphasize that in dynamic scope the tree of environment frames has exactly the same shape as the invocation tree. You should study the environment diagram and justify the target of each parent pointer. Under dynamic scoping, the first invocation of $sub3$ (on 13) yields 1 because the $n$ used in the subtraction is the program parameter $n$ (which is 12) rather than the 3 used as an argument to $create$-$sub$ when creating $sub3$. The second invocation of $sub3$ (on 0) yields -1 because the $n$ found this time is the argument 1 to test. The invocation of $sub2$ (on -1) finds that $n$ is this same 1, and returns -2 as the final result of the program.

### 2.2 Interpreter Implementation of Dynamic Scope

The two rules of the dynamic scoping mechanism are easy to encode in the environment model. The implementation is shown in Figure 5. For the first rules, the evaluation of an abstraction just returns the abstraction. For the second rules, the application of a function passes the call-time environment to funapply-dynamic, where it is used as the parent of the environment frame created for the application.
2.3 Comparing Static and Dynamic Scope

SNOBOL4, APL, most early Lisp dialects, and many macro languages are dynamically scoped. In each of these languages, a free variable in a function (or macro) body gets its meaning from the environment at the point where the function is called rather than the environment at the point where the function is created. Thus, in these languages, it is not possible to determine a unique declaration corresponding to a given free variable reference; the effective declaration depends on where the function is called. It is therefore generally impossible to determine the scope of a declaration simply by considering the abstract syntax tree of the program.

By and large, however, most modern languages use static scoping because, in practice, static scoping is often preferable to dynamic scoping. There are several reasons for this:

- Static scoping has better modularity properties than dynamic scoping. In a statically scoped
Figure 7: Essence of dynamic scoping in HoFl.

language, the particular names chosen for variables in a function do not affect its behavior, so it is always safe to rename them in a consistent fashion. In contrast, in dynamically scoped systems, the particular names chosen for variables matter because a local variable name can interact with a free variable name of a function invoked in its scope. Function interfaces are more complex under dynamic scoping because they must mention the free variables of the function.

- Static scoping works nicely with block structure to create higher-order functions that “remember” information from outer scopes. Many of the functional-programming idioms we have studied depend critically on this “memory” to work properly. As a simple example, consider the HoFl definition

\[
\text{(def add (abs x (abs y (+ x y)))}
\]

Under static scope, (\text{add 1}) stands for an incrementing function because the returned function “remembers” that \(x\) is 1. But under dynamic scope, (\text{add 1}) “forgets” that \(x\) is 1. The returned function is equivalent to (\text{abs y (+ x y)}) and will use whatever value for \(x\) it finds (if there is one) in the context where it is called. Clearly, dynamic scope and higher-order functions do not mix well!

In particular, in HoFl, multi-parameter functions are desugared into single-parameter functions using currying, whose proper behavior depends critically on the sort of “remembering” described above. So multi-parameter functions will simply not work as expected in a dynamically-scoped version of HoFl! For this reason, multi-parameter functions must be kernel constructs in a dynamically scoped language.

- Statically scoped variables can be implemented more efficiently than dynamically scoped variables. In a compiler, references to statically scoped variables can be compiled to code that accesses the variable value efficiently using its \textit{lexical address}, a description of its location that can be calculated from the program’s abstract syntax tree. In contrast, looking up dynamically scoped variables implies an inefficient search through a chain of bindings for one that has the desired name.

Is dynamic scoping ever useful? Yes! There are at least two situations in which dynamic scoping is important:

- \textit{Exception Handling}: In the languages we have studied so far, computations cannot proceed after encountering an error. However, later we will study ways to specify so-called \textit{exception handlers} that describe how a computation can proceed from certain kinds of errors.
Since exception handlers are typically in effect for certain subtrees of a program’s execution tree, dynamic scope is the most natural scoping mechanism for the namespace of exception handlers.

- **Implicit Parameters**: Dynamic scope is also convenient for specifying the values of implicit parameters that are cumbersome to list explicitly as formal parameters to functions. For example, consider the following `derivative` function in a version of HOFL with floating point operations (prefixed with `fp`):

```lisp
(def derivative
  (fun (f x)
       (fp/ (fp- (f (fp+ x epsilon))
             (f x))
             epsilon)))
```

Note that `epsilon` appears as a free variable in `derivative`. With dynamic scoping, it is possible to dynamically specify the value of `epsilon` via any binding construct. For example, the expression

```lisp
(bind epsilon 0.001
     (derivative (abs x (fp* x x)) 5.0))
```

would evaluate `(derivative (abs x (fp* x x)) 5.0)` in an environment where `epsilon` is bound to 0.001.

However, with lexical scoping, the variable `epsilon` must be defined at top level, and, without using mutation, there is no way to temporarily change the value of `epsilon` while the program is running. If we really want to abstract over `epsilon` with lexical scoping, we must pass it to `derivative` as an explicit argument:

```lisp
(def derivative
  (fun (f x epsilon)
       (fp/ (fp- (f (fp+ x epsilon))
              (f x))
              epsilon)))
```

But then any procedure that uses `derivative` and wants to abstract over `epsilon` must also include `epsilon` as a formal parameter. In the case of `derivative`, this is only a small inconvenience. But in a system with a large number of tweakable parameters, the desire for fine-grained specification of variables like `epsilon` can lead to an explosion in the number of formal parameters throughout a program.

As an example along these lines, consider the huge parameter space of a typical graphics system (colors, fonts, stippling patterns, line thicknesses, etc.). It is untenable to specify each of these as a formal parameter to every graphics routine. At the very least, all these parameters can be bundled up into a data structure that represents the graphics state. But then we still want a means of executing window routines in a temporary graphics state in such a way that the old graphics state is restored when the routines are done. Dynamic scoping is one technique for achieving this effect; side effects are another (as we shall see later).

### 3 Recursive Bindings

#### 3.1 The `bindrec` Construct

HOFL’s `bindrec` construct allows creating mutually recursive structures. For example, here is the classic `even?/odd?` mutual recursion example expressed in HOFL:
The scope of the names bound by bindrec (even? and odd? in this case) includes not only the body of the bindrec expression, but also the definition expressions bound to the names. This distinguishes bindrec from bindpar, where the scope of the names would include the body, but not the definitions. The difference between the scoping of bindrec and bindpar can be seen in the two contour diagrams in Fig. 8. In the bindrec expression, the reference occurrence of odd? within the even? abstraction has the binding name odd? as its binding occurrence; the case is similar for even?. However, when bindrec is changed to bindpar in this program, the names odd? and even? within the definitions become unbound variables. If bindrec were changed to bindseq, the occurrence of even? in the second binding would reference the declaration of even? in the first, but the occurrence of odd? in the first binding would still be unbound.

3.2 Substitution-Model Evaluation of bindrec

The evaluation rule for bindrec in the substitution-model interpreter is shown in Fig. 9. Each recursive definition in a bindrec is replaced by a copy of the definition in which each reference to a bindrec-bound name is replaced by an expression that wraps the name in a new bindrec with the same bindings. This has the effect of propagating the recursive nature of the bindrec to each reference to a bindrec-bound name.

To see how this works in practice, consider the following version of the even?/odd? program from above that has a simpler body:

Suppose we introduce the abbreviation $E$ for the abstraction

and the abbreviation $O$ for the abstraction
Figure 8: Lexical contours for versions of the **even?/odd?** program using **bindrec** and **bindpar**.
and eval exp =
match exp with
  : | Bindrec(names,defns,body) ->
      eval (substAll (map (fun defn -> Bindrec(names,defns,defn))
                       defns)
             names
             body)

Figure 9: Evaluation of bindrec in the substitution-model interpreter.

(abs y
 (if (= y 0)
   #f
   (even? (- y 1))))

Then Fig. 10 shows the substitution model evaluation of the program on the input 3. Note how
the substitution that wraps each bindrec-bound name in a fresh bindrec allows the abstractions
for the recursive functions to be unwound one level at a time, giving rise to the desired behavior
for the recursive functions.

(hofl (n) (bindrec ((even? E) (odd? O)) (even? n))) run on [3]
; Here and below, assume a ‘‘smart’’ substitution that
; performs renaming only when variable capture is possible.
⇒ (bindrec ((even? E) (odd? O)) (even? 2))
⇒ ((abs (x) (if (= x 0) #t ((bindrec ((even? E) (odd? O)) odd?) (- x 1)))) 3)
⇒ (if (= 3 0) #t ((bindrec ((even? E) (odd? O)) odd?) (- 3 1)))
⇒ (if #f #t ((bindrec ((even? E) (odd? O)) odd?) (- 3 1)))
⇒ (bindrec ((even? E) (odd? O)) odd?) (- 3 1)
⇒ ((abs (y) (if (= y 0) #f ((bindrec ((even? E) (odd? O)) even?) (- y 1)))) 2)
⇒ (if (= 2 0) #f ((bindrec ((even? E) (odd? O)) even?) (- 2 1)))
⇒ (if #f #t ((bindrec ((even? E) (odd? O)) even?) (- 2 1)))
⇒ ((bindrec ((even? E) (odd? O)) even?) (- 2 1))
⇒ ((abs (x) (if (= x 0) #t ((bindrec ((even? E) (odd? O)) odd?) (- x 1)))) 1)
⇒ (if (= 1 0) #t ((bindrec ((even? E) (odd? O)) odd?) (- 1 1)))
⇒ (if #f #t ((bindrec ((even? E) (odd? O)) odd?) (- 1 1)))
⇒ ((bindrec ((even? E) (odd? O)) odd?) (- 1 1))
⇒ ((abs (y) (if (= y 0) #f ((bindrec ((even? E) (odd? O)) even?) (- y 1)))) 0)
⇒ (if (= 0 0) #f ((bindrec ((even? E) (odd? O)) even?) (- 0 1)))
⇒ (if #t #f ((bindrec ((even? E) (odd? O)) even?) (- 0 1)))
⇒ #f

Figure 10: Example evaluation involving bindrec in the substitution-model interpreter.

3.3 Environment-Model Evaluation of bindrec

3.3.1 High-level Model

How is bindrec handled in the environment model? We do it in three stages:
1. Create an empty environment frame that will contain the recursive bindings, and set its parent pointer to be the environment in which the bindrec expression is evaluated.

2. Evaluate each of the definition expressions with respect to the empty environment. If evaluating any of the definition expressions requires the value of one of the recursively bound variables, the evaluation process is said to encounter a black hole and the bindrec is considered ill-defined.

3. Populate the new frame with bindings between the binding names and the values computed in step 2. Adding the bindings effectively “ties the knot” of recursion by making cycles in the graph structure of the environment diagram.

The result of this process for the even?/odd? example is shown below, where it is assumed that the program is called on the argument 5. The body of the program would be evaluated in environment ENV₁ constructed by the bindrec expression. Since the environment frames for containing x and y would all have ENV₁ as their parent pointer, the references to odd? and even? in these environments would be well-defined.

![Environment Diagram]

In order for bindrec to be meaningful, the definition expressions cannot require immediate evaluation of the bindrec-bound variables (else a black hole would be encountered). For example, the following bindrec example clearly doesn’t work because in the process of determining the value of x, the value x is required before it has been determined:

```
(bindrec ((x (+ x 1)))
  (* x 2))
```

In contrast, in the even?/odd? example we are not asking for the values of even? and odd? in the process of evaluating the definitions. Rather the definitions are abstractions that will refer to even? and odd? at a later time, when they are invoked. Abstractions serve as a sort of delaying mechanism that make the recursive bindings sensible.

As a more subtle example of a meaningless bindrec, consider the following:

```
(bindrec ((a (prep 1 b))
  (b (prep 2 a)))
  b)
```

Unlike the above case, here we can imagine that the definition might mean something sensible. Indeed in so-called call-by-need (a.k.a lazy) languages (such as Haskell), definitions like the above are very sensible, and stand for the following list structure:
However, call-by-value (a.k.a. strict or eager) languages (such as HOFL, OCAML, SCHEME, JAVA, C, etc.) require that all definitions be completely evaluated to values before they can be bound to a name or inserted in a data structure. In this class of languages, the attempt to evaluate (preps 1 b) fails because the value of b cannot be determined.

Nevertheless, by using the delaying power of abstractions, we can get something close to the above cyclic structure in HOFL. In the following program, the references to the recursive bindings one-two and two-one are “protected” within abstractions of zero variables (which are known as thunks). Any attempt to use the delayed variables requires applying the thunks to zero arguments (as in the expression ((snd stream)) within the prefix function).

\[
\text{hofl (n)} \\
\quad \text{(bindpar ((pair (fun (a b) (list a b)))}} \\
\quad \text{(fst (fun (pair) (head pair))))} \\
\quad \text{(snd (fun (pair) (head (tail pair))))}} \\
\quad \text{(bindrec ((one-two (pair 1 (fun () two-one)))}} \\
\quad \text{(two-one (pair 2 (fun () one-two)))}} \\
\quad \text{(prefix (fun (num stream) \text{\begin{align*}
\text{if} &\text{ (= num 0)} \\
\text{\quad (empty)} \\
\text{\quad (prep (fst stream) \\
\text{\quad \quad (prefix (- num 1) \text{((snd stream))}))}} \\
\end{align*}})))}} \\
\quad \text{(prefix n one-two)))}}
\]

When the above program is applied to the input 5, the result is (list 1 2 1 2 1).

### 3.3.2 Implementing `bindrec`

Implementing the “knot-tying” aspect of the recursive bindings of bindrec within the eval function of the statically scoped HOFL interpreter proves to be rather tricky. We will consider a sequence of incorrect definitions for the bindrec clause on the path to developing some correct ones.

Here is a first attempt:

\[
\begin{align*}
\text{(* Broken Attempt 1 *)} \\
\mid \text{Bindrec(names,defns,body)} \to \text{eval body} \\
\quad \text{(Env.bindAll names} \\
\quad \text{(map (fun defn \to eval defn ???)}} \\
\quad \text{defns)} \\
\quad \text{env)}
\end{align*}
\]

There is a problem here: what should the environment ??? be? It shouldn’t be env but the new environment that results from extending env with the recursive bindings. But the new environment has no name in the above clause!

A second attempt uses OCAML’s let to name the result of Env.bindAll:
(* Broken Attempt 2 *)
| Bindrec(names,defns,body) ->
| eval body
|      let newEnv = Env.bindAll names
|          (map (fun defn -> eval defn newEnv)
|                  defns)
|          env
|    in newEnv

This attempt fails because, by the scoping rules of Ocaml’s let construct, newEnv is an unbound variable in the expression Env.bindAll ... .

A third attempt replaces let with let rec:

(* Broken Attempt 3 *)
| Bindrec(names,defns,body) ->
| eval body
|      let rec newEnv =
|          Env.bindAll names
|          (map (fun defn -> eval defn newEnv)
|                  defns)
|          env
|    in newEnv

The above clause attempts to use the knot-tying ability of Ocaml’s own recursive binding construct, let rec, to implement Hofl’s recursive binding construct. Now the newEnv within Env.bindAll ... is indeed correctly scoped. Unfortunately, there are still two problems:

1. The let rec Syntax Restriction: The Ocaml let rec construct can only be used to define recursive functions that are specified by manifest abstractions. It cannot be used to define more general recursive values (such as recursive defined lists in the one-two example above).

2. The Call-by-Value Knot-Tying Problem: Even if Ocaml did allow more general recursive values to be defined via let rec, because Ocaml is a call-by-value language, we would still come face to face with the same sort of problem encountered in the recursive list example from above. That is, the eval within the functional argument to map requires that all its arguments be evaluated to values before it is invoked. But its newEnv argument is defined to be the result of a computation that depends on the result returned by invocations of this occurrence of eval. This leads to an irresolvable set of constraints: eval must return before it can be invoked!

We can fix the call-by-value knot-tying problem in the same way we fixed the recursive list problem above: by using thunks to delay evaluation of the recursive bound variable. In particular, rather than storing the result of evaluating the definition in the environment, we can store in the environment a thunk for evaluating the definition:

(* Broken Attempt 4 *)
| Bindrec(names,defns,body) ->
| eval body
|      let rec newEnv =
|          Env.bindAll names
|          (map (fun defn -> (fun () -> eval defn newEnv))
|                  defns)
|          env
|    in newEnv

This evaluation rule for bindrec is still broken because of the syntactic restrictions on let rec.²

²In call-by-value languages supporting more flexible recursive binding constructs, such as Scheme’s letrec, the
We'll see below that we can fix this problem as well. Once we do, we must also change the rest of the interpreter to ensure (1) that all entities stored in the environments used by \texttt{eval} are thunks and (2) that whenever a thunk is looked up in the environment, it should be “dethunked” - i.e., applied to zero arguments to retrieve its value. This makes sense if you think in terms of types. Point (1) says that the type of environments is effectively changed from \texttt{var -> valu} to \texttt{var -> unit -> value}, where \texttt{unit} is the type of \texttt{()}. Point (2) says that since the result of an environment lookup is now a function of type \texttt{unit -> valu}, it must be applied to zero arguments in order to get a value. Fig. 11 shows how these changes can be made in the \texttt{HOFL} environment-model interpreter.

\begin{verbatim}
(* 1. The definition of closure values must change, since they
    have an environment component: *)
and valu =
    :
    | Fun of var * exp * (unit -> valu) Env.env (* used to be valu Env.env *)

(* 2. The initial environment for evaluating the program body must be modified
    to thunk the argument integers. *)
let rec run (Pgm(fmls,body)) ints =
    :
    eval body (Env.make fmls (map (fun i -> fun () -> Int i)
        (* used to be (fun i -> Int i) *)
        ints))

(* 3. In function application, the new environment frame must use thunks: *)
and apply fcn arg =
    match fcn with
    Fun(fml,body,env) ->
        eval body (Env.bind fml (fun () -> arg) (* used to be arg *) env)
    | _ -> raise (EvalError("Non-function rator in application: " ^ (valuToString fcn)))

(* 4. When a variable reference is evaluated, it must dethunk the found thunk: *)
and eval exp env =
    match exp with
    :
    | Var name ->
        (match Env.lookup name env with
            Some(thunk) -> thunk () (* dethunk the found thunk *)
            | None -> raise (EvalError("Unbound variable: " ^ name)))

Figure 11: Changing the \texttt{HOFL} environment-model interpreter so that environments that bind
names to thunks rather than values.

To get around OCAML’s syntactic restrictions on \texttt{let rec}, we (1) need a way to convert between environments and functions and (2) need to change the \texttt{newEnv} definition to be a function definition. The \texttt{ENV} signature in Fig. 12 shows an environment signature that includes functions (\texttt{fromFun} and \texttt{toFun}) for converting between lookup functions and environments. These functions would be challenging to implement for list-based environment representations, but are trivial in the function-based environment representation used in the \texttt{Env} structure in Fig. 12. Using these conversion functions, we define the following working version of the \texttt{bindrec} evaluation rule:

fourth attempt at the \texttt{bindrec} evaluation rule works without further modification.
(* Working (but complex) Attempt 5 that: 
(1) thunks environment values to perform call-by-value knot-tying and 
(2) converts between environments and lookup functions to 
    get around OCAML's syntactic restrictions on LET REC. *)

| Bindrec(names,defns,body) ->
  eval body
  (let rec newEnvFcn =
    fun name ->
      (Env.toFun
        (Env.bindAll names
          (map (fun defn ->
            fun () ->
              eval defn (Env.fromFun newEnvFcn)))
          defns)
          name
        in Env.fromFun newEnvFcn)

This version defines a recursive lookup function newEnvFcn that takes a variable name and returns the result of looking it up in the recursive environment. The expression Env.fromFun newEnvFcn is used in the two spots where an environment is required. The environment resulting from Env.bindAll must be converted to a lookup function by Env.toFun so that it can be applied to the name argument. Conceptually, fun name -> E_{fcn} name is equivalent to E_{fcn} for any function-denoting expression E_{fcn}, but OCAML's let rec will not allow an arbitrary function-denoting expression for a definition; every such definition must be an abstraction.

Although the above approach "works", its technical acrobatics make it less elegant and less efficient than we'd like. A solution that is both more elegant and more efficient is possible if we utilize some other features of the environment signature and implementation in Fig. 12. This environment supports operations for binding both regular values (bind and bindAll) as well as thunked values (bindThunk and bindAllThunks) in such a way that the regular values may be looked up efficiently without any dethunking. Furthermore, it supports a abstract fixed point operator, fix, that internally uses OCAML's let rec construct to find the fixed point. This is possible because the implementation uses functions to represent environments, and let rec can be used to define recursive functions.

Using the extended environment module, we can implement bindrec as follows:

    (* Final Working Version *)
    Bindrec(names,defns,body) ->
      eval body
      (Env.fix (fun e ->
        (Env.bindAllThunks names
          (map (fun defn ->
            fun () ->
              eval defn ( Env.fromFun newEnvFcn)))
          defns)
          env))

This version does not require any other changes to the HOFL interpreter. For example, we would not make the changes in Fig. 11 with this version of the bindrec rule.

3.4 Fixed Points and the Y Operator

With all the complexity surrounding the implementation of bindrec in HOFL, it may be surprising that it is possible to define recursive functions in HOFL without bindrec! While bindrec is convenient for defining recursive functions, we will now see that it is not necessary. Higher-order functions have the power to express recursive computational processes by themselves.
module type ENV = sig
  type 'a env
  val empty : 'a env
  val bind : string -> 'a -> 'a env -> 'a env
  val bindAll : string list -> 'a list -> 'a env -> 'a env
  val make : string list -> 'a list -> 'a env
  val lookup : string -> 'a env -> 'a option
  val bindThunk : string -> (unit -> 'a) -> 'a env -> 'a env
  val bindAllThunks : string list -> (unit -> 'a) list -> 'a env -> 'a env
  val merge : 'a env -> 'a env -> 'a env
  val fix : ('a env -> 'a env) -> 'a env
  (* for converting between lookup functions and environments *)
  val fromFun : (string -> 'a option) -> 'a env
  val toFun : 'a env -> (string -> 'a option)
end

module Env : ENV = struct
  type 'a env = string -> 'a option

  let empty = fun s -> None

  let bind name valu env =
    fun s -> if s = name then Some valu else env s

  let bindAll names vals env = ListUtils.foldr2 bind env names vals

  let make names vals = bindAll names vals empty

  let lookup name env = env name

  (* val bindThunk : string -> (unit -> 'a) -> 'a env -> 'a env *)
  let bindThunk name valuThunk env =
    fun s -> if s = name then Some (valuThunk ()) else env s

  (* val bindAllThunks : string list -> (unit -> 'a) list -> 'a env -> 'a env *)
  let bindAllThunks names valThunks env =
    ListUtils.foldr2 bindThunk env names valThunks

  let merge env1 env2 =
    fun s -> (match env1 s with
      None -> env2 s
      | some -> some)

  (* val fix : ('a env -> 'a env) -> 'a env *)
  let fix gen = (* assume gen has type ('a env -> 'a env) -> 'a env *)
    (* define a recursive environment envfix, which is a function from
     a string name to a value of type 'a *)
    let rec envfix name = (gen envfix) name
    in envfix

  let fromFun f = f

  let toFun f = f
end

Figure 12: An environment signature and an function-based implementation of this signature.
A key step is converting a recursive definition for a function \( f \) to a so-called generating function \( g \) that maps functions to functions and has \( f \) as its fixed point — that is, \((gf) = f\). For example, here is a generating function (expressed in HOFL) for the factorial function:

\[
\text{(def fact-gen (fun (f))}
\begin{align*}
&\text{ (fun (n))} \\
&\quad \text{ (if (= n 0) \)}
&\quad \text{ 1} \\
&\quad \text{ (* n (f (- n 1))))))}
\end{align*}
\]

You should convince yourself that the factorial function is the only fixed point of \text{fact-gen}.

Amazingly, we can define a recursionless function that automatically finds the fixed points of generating functions like \text{fact-gen}. It is called the \text{Y operator}. Here is a HOFL definition of the \text{Y operator}:

\[
\text{(def y (fun (g))}
\begin{align*}
&\text{ ((fun (s) (fun (x) ((g (s s)) x)))} \\
&\text{ (fun (s) (fun (x) ((g (s s)) x))))))}
\end{align*}
\]

Using \( y \), we can define the factorial function as

\[
\text{(def fact (y fact-gen))}
\]

Note that no recursion has been used!

To see how this work, let’s use the abbreviation \( FG \) for

\[
\begin{align*}
&\text{(fun (f))} \\
&\quad \text{ (fun (n))} \\
&\quad \quad \text{ (if (= n 0) \)} \\
&\quad \quad \quad \text{ 1} \\
&\quad \quad \quad \text{ (* n (f (- n 1)))))}
\end{align*}
\]

and the abbreviation \( FS \) for

\[
\begin{align*}
&\text{(fun (s) (fun (x) ((FG (s s)) x)))}
\end{align*}
\]

Note that in the substitution model,

\[
(FS FS) \Rightarrow (fun (x) ((FG (FS FS)) x))
\]

Now we’re ready to see how \text{fact} works in an example:

\[
\begin{align*}
&((y \ FG) 3) \\
&\Rightarrow ((FS FS) 3) \\
&\Rightarrow ((fun (x) ((FG (FS FS) x))) 3) \\
&\Rightarrow ((FG (FS FS) 3)) \\
&\Rightarrow ((FG (FS FS) 3))
\end{align*}
\]"
We conclude by showing that we can even define mutually recursive functions using \( y \) by using Church pairs to pair the functions:

\[
\text{(def church-pair (fun (x y) (fun (f) (f x y))))}
\]

\[
\text{(def church-fst (fun (p) (p (fun (x y) x))))}
\]

\[
\text{(def church-snd (fun (p) (p (fun (x y) y))))}
\]

\[
\text{(def even-odd-gen (fun (p)}
\text{  (church-pair}
\text{    (fun (x) ; even?}
\text{      (if (= x 0)
\text{        #t
\text{          ((church-snd p) (- x 1))))
\text{      (fun (y) ; odd?
\text{        (if (= y 0)
\text{          #f
\text{            ((church-fst p) (- y 1)))))
\text{    ))))}
\text{  ))))}
\]

\[
\text{(def even? (church-fst (y even-odd-gen)))}
\]

\[
\text{(def odd? (church-snd (y even-odd-gen)))}
\]