The Pros of `cons`: Programming with Pairs and Lists

CS251 Programming Languages
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Racket Values

- booleans: `#t, #f`
- numbers:
  - integers: `42, 0, -273`
  - rationals: `2/3, -251/17`
  - floating point (including scientific notation): `98.6, -6.125, 3.141592653589793, 6.023e23`
  - complex: `3+2i, 17-23i, 4.5-1.4142i`

  Note: some are *exact*, the rest are *inexact*. See docs.
- strings: `"cat"`, `"CS251"`, `"αβγ"`, `"To be\nor not\nto be"`
- characters: `#\a`, `#\A`, `#\5`, `#\space`, `#\tab`, `#\newline`
- anonymous functions: `(lambda (a b) (+ a (* b c)))`

What about compound data?

`cons` Glues Two Values into a Pair

A new kind of value:

- pairs (a.k.a. `cons` cells): `(cons` ***V1 V2***`)`
  e.g.,
  - `(cons 17 42)`
  - `(cons 3.14159 #t)`
  - `(cons "CS251" (λ (x) (* 2 x))`
  - `(cons (cons 3 4.5) (cons #f #\a))`
- Can glue any number of values into a `cons` tree!

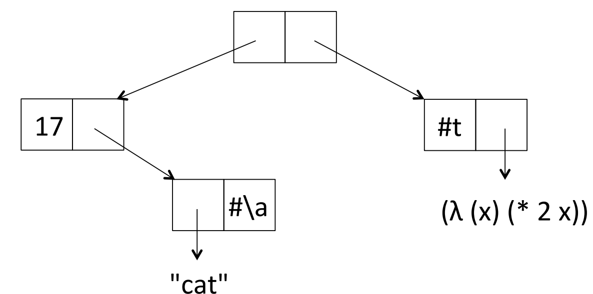
In Racket, type Command-\ to get λ char

Box-and-pointer diagrams for `cons` trees

`(cons v1 v2)` | ***v1*** | ***v2*** |

Convention: put "small" values (numbers, booleans, characters) inside a box, and draw a pointers to "large" values (functions, strings, pairs) outside a box.

```
(cons (cons 17 (cons "cat" #\a))
      (cons #t (λ (x) (* 2 x))))
```



Evaluation Rules for `cons`

Big step semantics:

$$\frac{E1 \downarrow V1 \quad E2 \downarrow V2}{\texttt{(cons } E1\ E2\texttt{)} \downarrow \texttt{(cons } V1\ V2\texttt{)}}\ [\text{cons}]$$

Small-step semantics:

cons has no special evaluation rules. Its two operands are evaluated left-to-right until a value **(cons *V1 V2*)** is reached:

(cons *E1 E2*)
⇒* **(cons *V1 {E2}*)** ; first evaluate ***E1*** to ***V1*** step-by-step
⇒* **(cons *V1 V2*)** ; then evaluate ***e2*** to ***v2*** step-by-step

`cons` evaluation example

```
(cons (cons {(+ 1 2)} (< 3 4))
      (cons (> 5 6) (* 7 8)))
⇒ (cons (cons 3 {(< 3 4)})
        (cons (> 5 6) (* 7 8)))
⇒ (cons (cons 3 #t) (cons {(> 5 6)} (* 7 8)))
⇒ (cons (cons 3 #t) (cons #f {(* 7 8)}))
⇒ (cons (cons 3 #t) (cons #f 56))
```


`car` and `cdr`

- `car` extracts the left value of a pair

  `(car (cons 7 4))` ⇒ `7`

- `cdr` extract the right value of a pair

  `(cdr (cons 7 4))` ⇒ `4`

Why these names?

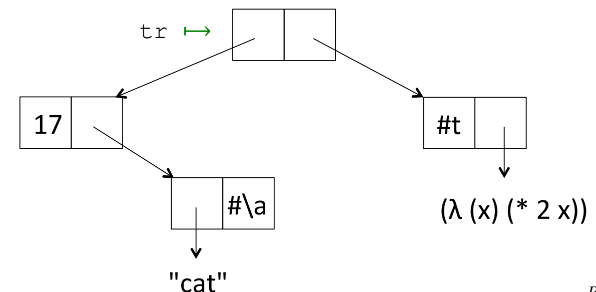
- `car` from "contents of address register"
- `cdr` from "contents of decrement register"

Practice with `car` and `cdr`

Write expressions using `car`, `cdr`, and `tr` that extract the five leaves of this tree:

```
(define tr (cons (cons 17 (cons "cat" #\a))
                 (cons #t (λ (x) (* 2 x)))))
```

```
tr ↦ (cons (cons 17 (cons "cat" #\a))
           (cons #t (λ (x) (* 2 x)))), …
```



`cadr` and friends

- `(caar e)` means `(car (car e))`
- `(cadr e)` means `(car (cdr e))`
- `(cdar e)` means `(cdr (car e))`
- `(cddr e)` means `(cdr (cdr e))`
- `(caaar e)` means `(car (car (car e)))`

  ⋮
- `(cddddr e)` means `(cdr (cdr (cdr (cdr e))))`

Evaluation Rules for `car` and `cdr`

Big-step semantics:

$$\frac{E \downarrow (\texttt{cons}\ V1\ V2)}{(\texttt{car}\ E) \downarrow V1}\ [\text{car}] \qquad \frac{E \downarrow (\texttt{cons}\ V1\ V2)}{(\texttt{cdr}\ e) \downarrow v2}\ [\text{cdr}]$$

Small-step semantics:

$(\texttt{car}\ (\texttt{cons}\ V1\ V2)) \Rightarrow V1$ [car]

$(\texttt{cdr}\ (\texttt{cons}\ V1\ V2)) \Rightarrow V2$ [cdr]

Semantics Puzzle

According to the rules on the previous page, what is the result of evaluating this expression?

```
(car (cons (+ 2 3) (* 5 #t)))
```

Note: there are two ``natural" answers. Racket gives one, but there are languages that give the other one!

Printed Representations in Racket Interpreter

```
> (lambda (x) (* x 2))
#<procedure>

> (cons (+ 1 2) (* 3 4))
'(3 . 12)

> (cons (cons 5 6) (cons 7 8))
'((5 . 6) 7 . 8)

> (cons 1 (cons 2 (cons 3 4)))
'(1 2 3 . 4)
```

What's going on here?

Display Notation and Dotted Pairs

- The **display notation** for `(cons` ***V1 V2***`)` is `(`***DN1*** `.` ***DN2***`)`, where ***DN1*** and ***DN2*** are the display notations for ***V1*** and ***V2***
- In display notation, a dot "eats" a paren pair that follows it directly:

  `((5 . 6) . (7 . 8))`
    becomes `((5 . 6) 7 . 8)`

  `(1 . (2 . (3 . 4)))`
    becomes `(1 . (2 3 . 4))`
    becomes `(1 2 3 . 4)`

  Why? Because we'll see this makes lists print prettily.
- The Racket interpreter puts a single quote mark before the display notation of a top-level pair value. (We'll say more about quotation later.)

`display` vs. `print` in Racket

```
> (display (cons 1 (cons 2 null)))
(1 2)

> (display (cons (cons 5 6) (cons 7 8)))
((5 . 6) 7 . 8)

> (display (cons 1 (cons 2 (cons 3 4))))
(1 2 3 . 4)
```

```
> (print(cons 1 (cons 2 null)))
'(1 2)

> (print(cons (cons 5 6) (cons 7 8)))
'((5 . 6) 7 . 8)

> (print(cons 1 (cons 2 (cons 3 4))))
'(1 2 3 . 4)
```


Functions Can Take and Return Pairs

```
(define (swap-pair pair)
   (cons (cdr pair) (car pair)))

(define (sort-pair pair)
   (if (< (car pair) (cdr pair))
       pair
       (swap pair)))
```

What are the values of these expressions?

- `(swap-pair (cons 1 2))`
- `(sort-pair (cons 4 7))`
- `(sort-pair (cons 8 5))`

Lists

In Racket, a **list** is just a recursive pattern of pairs.

A list is either

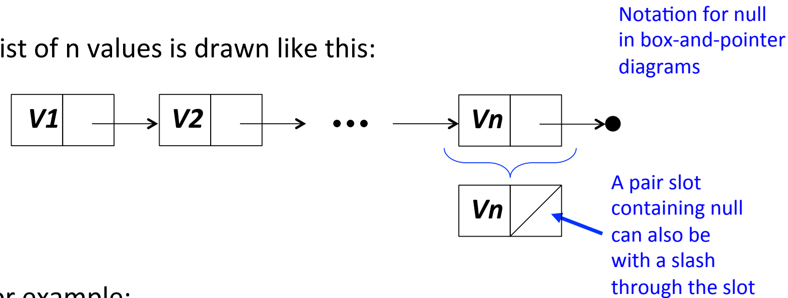
- The empty list `null`, whose display notation is ()
- A nonempty list `(cons` ***Vfirst Vrest***`)` whose
  - first element is ***Vfirst***
  - and the rest of whose elements are the sublist ***Vrest***

E.g., a list of the 3 numbers 7, 2, 4 is written

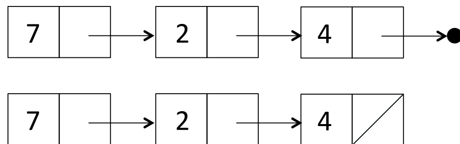
`(cons 7 (cons 2 (cons 4 null)))`

Box-and-pointer notation for lists

A list of n values is drawn like this:



For example:



`list` sugar

Treat `list` as syntactic sugar:

- `(list)` desugars to `null`
- `(list` ***E1*** `…)` desugars to `(cons` ***E1*** `(list …))`

For example:

```
(list (+ 1 2) (* 3 4) (< 5 6))
desugars to (cons (+ 1 2) (list (* 3 4) (< 5 6)))
desugars to (cons (+ 1 2) (cons (* 3 4) (list (< 5 6))))
desugars to (cons (+ 1 2) (cons (* 3 4) (cons (< 5 6) (list))))
desugars to (cons (+ 1 2) (cons (* 3 4) (cons (< 5 6) null)))
```

* This is a white lie, but we can pretend it's true for now

Display Notation for Lists

The "dot eats parens" rule makes lists display nicely:

```
(list 7 2 4)
desugars to (cons 7 (cons 2 (cons 4 null))))
displays as (before rule) (7 . (2 . (4 . ())))
displays as (after rule) (7 2 4)
prints as '(7 2 4)
```

In Racket:

```
> (display (list 7 2 4))
(7 2 4)

> (display (cons 7 (cons 2 (cons 4 null))))
(7 2 4)
```


`list` and small-step evaluation

It is sometimes helpful to both desugar and resugar with `list`:

```
(list (+ 1 2) (* 3 4) (< 5 6))
desugars to (cons {(+ 1 2)} (cons (* 3 4)
                                   (cons (< 5 6) null)))
⇒ (cons 3 (cons {(* 3 4)} (cons (< 5 6) null)))
⇒ (cons 3 (cons 12 (cons {(< 5 6)} null)))
⇒ (cons 3 (cons 12 (cons #t null)))
resugars to (list 3 12 #t)
```

Heck, let's just informally write this as:

```
(list {(+ 1 2)} (* 3 4) (< 5 6))
⇒ (list 3 {(* 3 4)} (< 5 6))
⇒ (list 3 12 {(< 5 6)})
⇒ (list 3 12 #t)
```

`first`, `rest`, and friends

- `first` returns the first element of a list:
  `(first (list 7 2 4))` ⇒ `7`
  (`first` is almost a synonym for `car`, but requires its argument to be a list)
- `rest` returns the sublist of a list containing every element but the first:
  `(rest (list 7 2 4))` ⇒ `(list 2 4)`
  (`rest` is almost a synonym for `cdr`, but requires its argument to be a list)
- Also have `second`, `third`, …, `ninth`, `tenth`

Recursive List Functions

Because lists are defined recursively, it's natural to process them recursively.

Typically (but not always) a recursive function `recf` on a list argument `L` has two cases:

- **base case:** what does `recf` return when L is empty? (Use `null?` to test for an empty list)
- **recursive case:** if L is the nonempty list `(cons` ***Vfirst Vrest***`)` how are ***Vfirst*** and `(recf` ***Vrest***`)` combined to give the result for `(recf L)`?

  Note that we **always** ``blindly" apply `recf` to ***Vrest***!

Recursive List Functions: Divide/Conquer/Glue (DCG) strategy for the general case [in words]

Step 1 (concrete example): pick a concrete input list, typically 3 or 4 elements long. What should the function return on this input?

E.g. A `sum` function that returns the sum of all the numbers in a list:
`(sum '(5 7 2 4))` ⇒* `18`

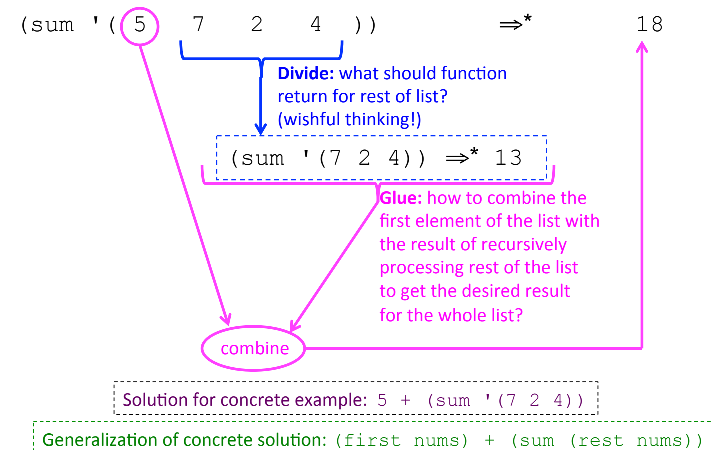
Step 2 (divide): without even thinking, **always** apply the function to the `rest` of the list. What does it return? `(sum '(7 2 4))` ⇒* `13`

Step 3 (glue): How to combine the first element of the list (in this case, `5`) with the result from processing the rest (in this case, `13`) to give the result for processing the whole list (in this case, `18`)? `5 + (sum '(7 2 4))` ⇒* `18`

Step 4 (generalize): Express the general case in terms of an arbitrary input:

```
(define (sum nums)
   … (+ (first nums) (sum (rest nums)) … )
```

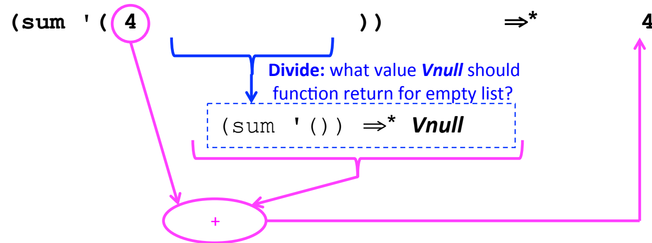

Recursive List Functions: Divide/Conquer/Glue (DCG) strategy for the general case [in diagram]



Recursive List Functions: base case via singleton case

Deciding what a recursive list function should return for the empty list is not always obvious and can be tricky. E.g. what should `(sum '())` return?

If the answer isn't obvious, consider the ``penultimate case" in the recursion, which involves a list of one element:



In this case, ***Vnull*** should be 0, which is the identity element for addition.

But in general it depends on the details of the particular combiner determined from the general case. So solve the general case before the base case!

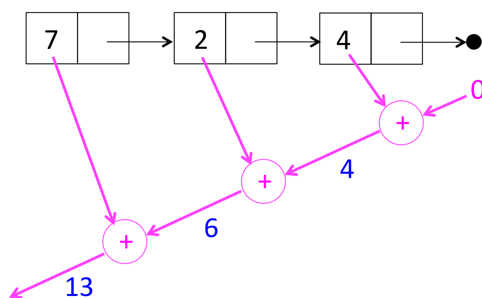
Putting it all together: base & general cases

`(sum nums)` returns the sum of the numbers in the list `nums`

```
(define (sum ns)
  (if (null? ns)
      0
      (+ (first ns)
         (sum (rest ns)))))
```


Understanding `sum`: Approach #1

`(sum '(7 2 4))`



We'll call this the **recursive accumulation** pattern

Understanding `sum`: Approach #2

In `(sum (list 7 2 4))`, the list argument to `sum` is

`(cons 7 (cons 2 (cons 4 null))))`

Replace `cons` by **+** and `null` by **0** and simplify:

`(+ 7 (+ 2 (+ 4 0))))`

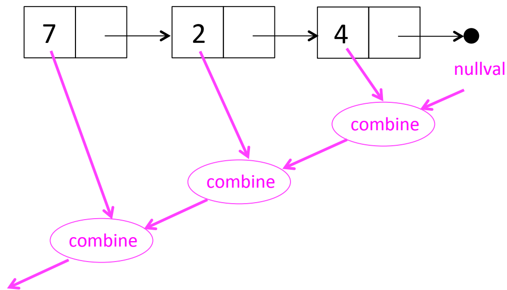
⇒ `(+ 7 (+ 2 4)))`

⇒ `(+ 7 6)`

⇒ **13**

Generalizing sum: Approach #1

```
(recf (list 7 2 4))
```



Generalizing sum: Approach #2

In `(recf (list 7 2 4))`, the list argument to `recf` is

```
(cons 7 (cons 2 (cons 4 null)))
```

Replace `cons` by **`combine`** and `null` by **`nullval`** and simplify:

```
(combine 7 (combine 2 (combine 4 nullval)))
```


Generalizing the sum definition

```
(define (recf ns)
  (if (null? ns)
      nullval
      (combine (first ns)
               (recf (rest ns)))))
```


Your turn

Define the following recursive list functions and test them in Racket:

`(product ns)` returns the product of the numbers in `ns`

`(min-list ns)` returns the minimum of the numbers in `ns`
Hint: use `min` and `+inf.0` (positive infinity)

`(max-list ns)` returns the minimum of the numbers in `ns`
Hint: use `max` and `-inf.0` (negative infinity)

`(all-true? bs)` returns `#t` if all the elements in `bs` are truthy; otherwise returns `#f`. *Hint*: use `and`

`(some-true? bs)` returns a truthy value if at least one element in `bs` is truthy; otherwise returns `#f`. Hint: use `or`

`(my-length xs)` returns the length of the list `xs`

Recursive Accumulation Pattern Summary

	combine	nullval
sum	+	0
product	*	1
min-list	min	+inf.0
max-list	max	-inf.0
all-true?	and	#t
some-true?	or	#f
my-length	(λ (fst subres) (+ 1 subres))	0

Mapping Example: `map-double`

`(map-double ns)` returns a new list the same length as `ns` in which each element is the double of the corresponding element in `ns`.

```
> (map-double (list 7 2 4))
'(14 4 8)
```

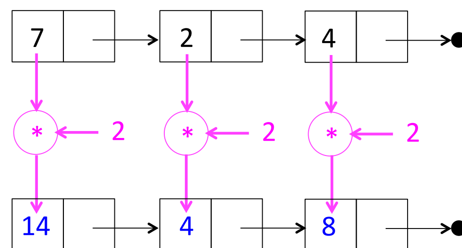
```
(define (map-double ns)
  (if (null? ns)
      ; Flesh out base case

      ; Flesh out general case

      ))
```


Understanding `map-double`

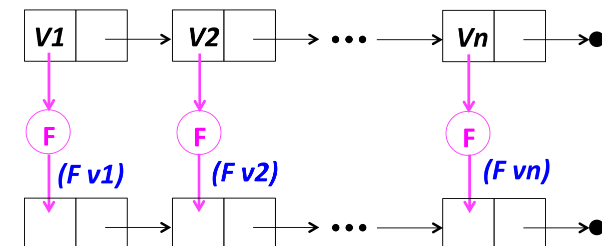
`(map-double '(7 2 4))`



We'll call this the **mapping** pattern

Generalizing `map-double`

`(mapF (list V1 V2 … Vn))`




```
(define (mapF xs)
  (if (null? xs)
      null
      (cons (F (first xs))
            (mapF (rest xs)))))
```

Expressing `mapF` as an accumulation

```
(define (mapF xs)
  (if (null? xs)
      null
      ((λ (fst subres)
                         ) ; Flesh this out
       (first xs)
       (mapF (rest xs)))))
```


Some Recursive Listfuns Need Extra Args

```
(define (map-scale factor ns)
  (if (null? ns)
      null
      (cons (* factor (first ns))
            (map-scale factor (rest ns)))))
```


Filtering Example: `filter-positive`

`(filter-positive ns)` returns a new list that contains only the positive elements in the list of numbers `ns`, in the same relative order as in `ns`.

```
> (filter-positive (list 7 -2 -4 8 5))
'(7 8 5)
```

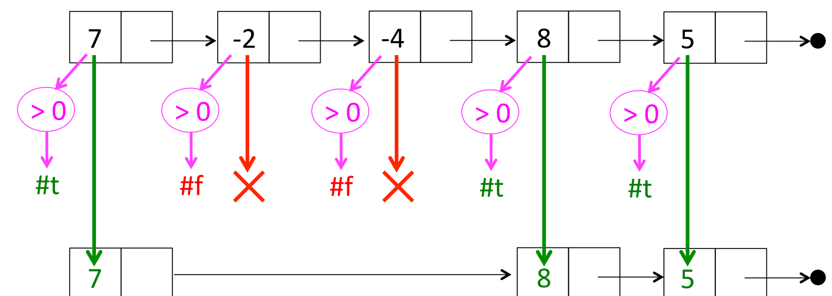
```
(define (filter-positive ns)
  (if (null? ns)
      ; Flesh out base case

      ; Flesh out recursive case

      ))
```


Understanding `filter-positive`

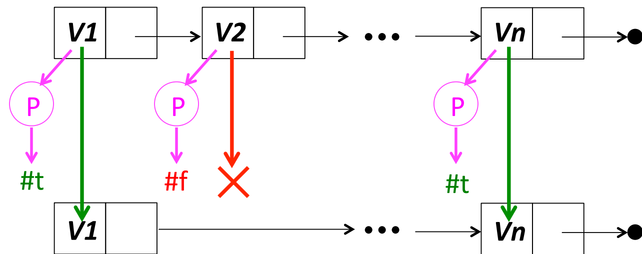
```
(filter-positive (list 7 -2 -4 8 5))
```



We'll call this the **filtering** pattern

Generalizing `filter-positive`

`(filterP (list V1 V2 … Vn))`




```
(define (filterP xs)
  (if (null? xs)
      null
      (if (P (first xs))
          (cons (first xs) (filterP (rest xs)))
          (filterP (rest xs)))))
```


Expressing `filterP` as an accumulation

```
(define (filterP xs)
  (if (null? xs)
      null
      ((lambda (fst subres)


                              ) ; Flesh this out
       (first xs)
       (filterP (rest xs)))))
```


Your turn:
Define these using Divide/Conquer/Glue

```
> (snoc 11 '(7 2 4))
'(7 2 4 11)

> (my-append '(7 2 4) '(5 8))
'(7 2 4 5 8)

> (append-all '((7 2 4) (9) () (5 8)))
'(7 2 4 9 5 8)

> (my-reverse '(5 7 2 4))
'(4 2 7 5)
```