More on Parametric Equations
Computer Graphics

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Parametric equations of lines are very cool, and most of what you remember from high-school math will serve you well. This document will help fill out some of the applications that we might want to do

1 Line Intersections

Graphics systems and graphics programmers often care about whether lines or line segments intersect. Let's look at one example. Here are four points:

$$\begin{aligned} A &= (0, 3, 1) \\ B &= (12, 7, 17) \\ C &= (2, 0, 8) \\ D &= (7, 20, -2) \end{aligned}$$

The two lines we are interested in are $P = AB$ and $Q = CD$. Each line gets its own parameter (why?), so the parametric equations are

$$\begin{aligned} P(t) &= A + (B - A)t \\ Q(s) &= C + (D - C)s \end{aligned}$$

We'll need the vectors on the RHS a lot, so let's name them and calculate them:

$$\begin{aligned} v &= B - A = (12, 7, 17) - (0, 3, 1) = (12, 4, 16) \\ w &= D - C = (7, 20, -2) - (2, 0, 8) = (5, 20, -10) \end{aligned}$$

Okay, now we're ready to go.

2 2D Line Intersections

To warm up a bit before we get to 3D, let's first find the intersection in 2D. We'll just have to ignore the z coordinate:

$$\begin{aligned} P(t) &= (0, 3) + (12, 4)t \\ Q(s) &= (2, 0) + (5, 20)s \end{aligned}$$

Now, at the intersection point, the point on $P(t)$ is equal to the point on $Q(s)$:

$$\begin{aligned} P(t) &= Q(s) \\ (0, 3) + (12, 4)t &= (2, 0) + (5, 20)s \end{aligned} \tag{1}$$

That last equation is just shorthand for two equations (one for each dimension: x and y) in two unknowns, s and t:

$$\begin{aligned} 0 + 12t &= 2 + 5s \\ 3 + 4t &= 0 + 20s \end{aligned} \quad (2)$$

Multiplying the second equation by three and subtracting it from the first yields:

$$\begin{aligned} -9 &= 2 + 5s - 60s \\ -11 &= -55s \\ s &= 1/5 = 0.2 \end{aligned}$$

If $s = 0.2$, we can substitute it into either of the equations above to find t. Actually, let's substitute it into both, to check that it works for both.

$$\begin{aligned} 0 + 12t &= 2 + 5(0.2) \Rightarrow t = 1/4 = 0.25 \\ 3 + 4t &= 0 + 20(0.2) \Rightarrow t = 1/4 = 0.25 \end{aligned}$$

So, using $s = 0.2$ and $t = 0.25$, we can compute the intersection point. Again, let's compute it twice, to double-check:

$$\begin{aligned} P(0.25) &= (0,3) + (12,4)0.25 \Rightarrow (0,3) + (3,1) \Rightarrow (3,4) \\ Q(0.20) &= (2,0) + (5,20)0.20 \Rightarrow (2,0) + (1,4) \Rightarrow (3,4) \end{aligned}$$

So, the intersection point is the same. Whew!

What else do we know? Since both s and t are between 0 and 1, we know that the line *segments* intersect, not just the lines. In fact, we know that the intersection point is exactly one quarter of the way from A to B and one fifth of the way from C to D.

Very cool! Take that, $y = mx + b$!

3 3D Line Intersections

What about doing the intersection using the 3D lines? Then, we get *three* equations in two unknowns (compare with equation 1).

$$\begin{aligned} P(t) &= Q(s) \\ (0,3,1) + (12,4,16)t &= (2,0,8) + (5,20,-10)s \end{aligned} \quad (3)$$

This is equivalent to three equalities, one for each dimension (x, y and z) in two unknowns, s and t. Compare to equation 2)

$$\begin{aligned} 0 + 12t &= 2 + 5s \\ 3 + 4t &= 0 + 20s \\ 1 + 16t &= 8 - 10s \end{aligned} \quad (4)$$

We can then choose any two of them to solve for s and t. If we choose the first two, the algebra is *identical*, so we don't have to re-do it here. However, we do need to check that the z equation is satisfied:

$$\begin{aligned} 1 + 16(0.25) &= 8 - 10(0.20) \\ 5 &= 6 \end{aligned}$$

Oops! So, the z equation is *not* satisfied.
Let's check what happens to the point of intersection:

$$\begin{aligned} P(0.25) &= (0,3,1) + (12,4,16)0.25 \Rightarrow (0,3,1) + (3,1,4) \Rightarrow (3,4,5) \\ Q(0.20) &= (2,0,8) + (5,20,-10)0.20 \Rightarrow (2,0,8) + (1,4,-2) \Rightarrow (3,4,6) \end{aligned}$$

So the 3D lines *don't* intersect. The x and y coordinates are the same, but the z coordinate is different. (The lines are *skew*.)

In general, lines in 2D almost always intersect. The only time they don't is when they are parallel. In 3D, lines almost never intersect.

4 Intersecting 3D Lines

They do sometimes, though. If we change points C and D, subtracting 1 from the z coordinate of each (this moves the line parallel to itself), we get two line segments that *do* intersect:

$$\begin{aligned} A &= (0,3,1) \\ B &= (12,7,17) \\ C &= (2,0,7) \\ D &= (7,20,-3) \end{aligned}$$

Here are the vectors. Note that they are the same as in the first problem, which shows that the new $Q(s)$ line is parallel to the old one:

$$\begin{aligned} v &= B - A = (12,7,17) - (0,3,1) = (12,4,16) \\ w &= D - C = (7,20,-3) - (2,0,7) = (5,20,-10) \end{aligned}$$

The equation we need to satisfy is nearly the same, except for the z coordinate of C:

$$\begin{aligned} P(t) &= Q(s) \\ (0,3,1) + (12,4,16)t &= (2,0,7) + (5,20,-10)s \end{aligned}$$

Now, the intersection parameters satisfy the z equation, too:

$$\begin{aligned} 1 + 16(0.25) &= 7 - 10(0.20) \\ 5 &= 5 \end{aligned}$$

And, the intersection points are really the same:

$$\begin{aligned} P(0.25) &= (0,3,1) + (12,4,16)0.25 \Rightarrow (0,3,1) + (3,1,4) \Rightarrow (3,4,5) \\ Q(0.20) &= (2,0,7) + (5,20,-10)0.20 \Rightarrow (2,0,7) + (1,4,-2) \Rightarrow (3,4,5) \end{aligned}$$