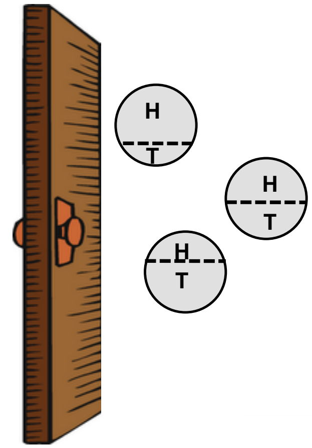
Hidden Markov Models

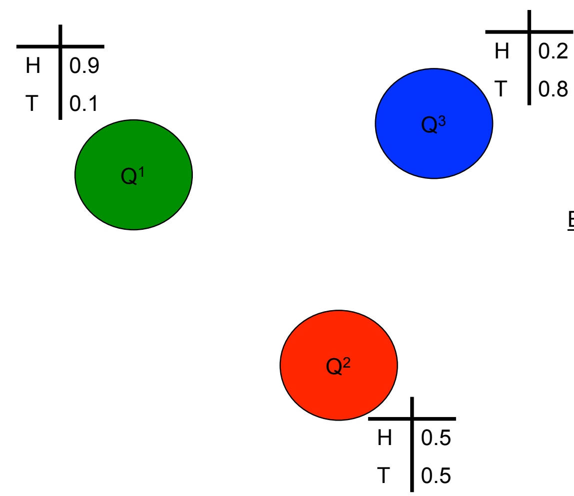
Coin Example



Coin Example

HTTHTTTHHT

HMM: Emission Probabilities, *B*



Emission Probabilities, *B*

$b_1(H) = 0.9$

$b_1(T) = 0.1$

$b_2(H) = 0.5$

$b_2(T) = 0.5$

$b_3(H) = 0.2$

$b_3(T) = 0.8$

Probability of Observation Sequence

If only state 1, i.e., the first coin, is used…

O **=**	**H**	**T**	**T**	**H**	**T**	**T**	**T**	**H**	**H**	**T**
	Q^1	Q^1	Q^1	Q^1	Q^1	Q^1	Q^1	Q^1	Q^1	Q^1
$P(O) =$	$b_1(H)$	$b_1(T)$	$b_1(T)$	$b_1(H)$	$b_1(T)$	$b_1(T)$	$b_1(T)$	$b_1(H)$	$b_1(H)$	$b_1(T)$
	0.9	0.1	0.1	0.9	0.1	0.1	0.1	0.9	0.9	0.1

0.0000006561

Probability of Observation Sequence

If only state 2, i.e., the second coin, is used…

O **=**	**H**	**T**	**T**	**H**	**T**	**T**	**T**	**H**	**H**	**T**
	Q^2	Q^2	Q^2	Q^2	Q^2	Q^2	Q^2	Q^2	Q^2	Q^2
$P(O) =$	$b_2(H)$	$b_2(T)$	$b_2(T)$	$b_2(H)$	$b_2(T)$	$b_2(T)$	$b_2(T)$	$b_2(H)$	$b_2(H)$	$b_2(T)$
	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

0.0009765625

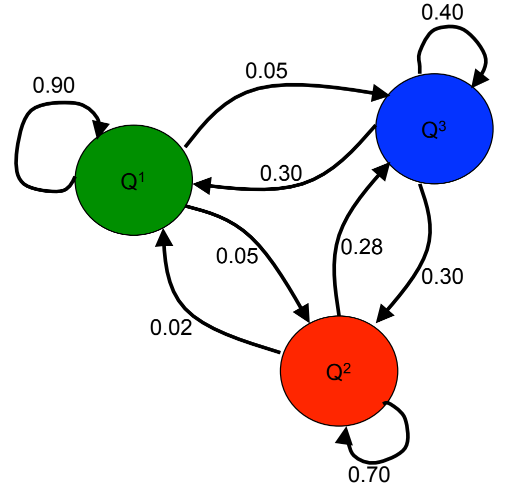
Probability of Observation Sequence

If only state 3, i.e., the third coin, is used…

O **=**	**H**	**T**	**T**	**H**	**T**	**T**	**T**	**H**	**H**	**T**
	Q^3	Q^3	Q^3	Q^3	Q^3	Q^3	Q^3	Q^3	Q^3	Q^3
$P(O) =$	$b_3(H)$	$b_3(T)$	$b_3(T)$	$b_3(H)$	$b_3(T)$	$b_3(T)$	$b_3(T)$	$b_3(H)$	$b_3(H)$	$b_3(T)$
	0.2	0.8	0.8	0.2	0.8	0.8	0.8	0.2	0.2	0.8

0.0004194304

HMM: Transition Probability, *A*



<u>Transition Probabilities, *A*</u>

$a_{11} = 0.90$
$a_{12} = 0.05$
$a_{13} = 0.05$

$a_{21} = 0.02$
$a_{22} = 0.70$
$a_{23} = 0.28$

$a_{31} = 0.30$
$a_{32} = 0.30$
$a_{33} = 0.40$

Probability of Observation Sequence

If we can transition between states (coins) ...

$O =$ H T T H T T T H H T

Q^1 Q^1 Q^1 Q^1 Q^3 Q^3 Q^3 Q^3 Q^3 Q^3

$P(O) =$ $b_1(H)a_{11}\, b_1(T)a_{11}\, b_1(T)a_{11}\, b_1(H)a_{13}\, b_3(T)a_{33}\, b_3(T)a_{33}\, b_3(T)a_{33}\, b_3(H)a_{33}\, b_3(H)a_{33}\, b_3(T)$

0.90 0.90 0.10 0.90 0.10 0.90 0.10 0.05 0.80 0.40 0.80 0.40 0.80 0.40 0.20 0.40 0.80 0.40 0.80

0.0000000220150628352

* Assuming we start in state 1, i.e., the first coin

Probability of Observation Sequence

If we can transition between states (coins) ...

$O =$ H T T H T T T H H T

Q^1 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2

$P(O) =$ $b_1(H)a_{12}\, b_2(T)a_{22}\, b_2(T)a_{22}\, b_2(H)a_{22}\, b_2(T)a_{22}\, b_2(T)a_{22}\, b_2(T)a_{22}\, b_2(H)a_{22}\, b_2(H)a_{22}\, b_2(T)$

0.90 0.05 0.50 0.70 0.50 0.70 0.50 0.70 0.50 0.70 0.50 0.70 0.50 0.70 0.50 0.70 0.50 0.70 0.50

0.0000050671962890525

* Assuming we start in state 1, i.e., the first coin

Probability of Observation Sequence

If we can transition between states (coins) ...

$O =$ H T T H T T T H H T

Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1

$P(O) =$ $b_1(H)a_{11}\, b_1(T)a_{11}\, b_1(T)a_{11}\, b_1(H)a_{11}\, b_1(T)a_{11}\, b_1(T)a_{11}\, b_1(T)a_{11}\, b_1(H)a_{11}\, b_1(H)a_{11}\, b_1(T)$

0.90 0.90 0.10 0.90 0.10 0.90 0.90 0.90 0.10 0.90 0.10 0.90 0.10 0.90 0.90 0.90 0.90 0.90 0.10

0.0000002541865828329

* Assuming we start in state 1, i.e., the first coin

Probability of Observation Sequence

If we can transition between states (coins) ...

$O =$ H T T H T T T H H T

Q^1 Q^3 Q^2 Q^2 Q^3 Q^3 Q^3 Q^1 Q^2 Q^2

$P(O) =$ $b_1(H)a_{13}\, b_3(T)a_{32}\, b_2(T)a_{22}\, b_2(H)a_{23}\, b_3(T)a_{33}\, b_3(T)a_{33}\, b_3(T)a_{31}\, b_1(H)a_{12}\, b_2(H)a_{22}\, b_2(T)$

0.90 0.05 0.80 0.30 0.50 0.70 0.50 0.28 0.80 0.40 0.80 0.40 0.80 0.30 0.90 0.05 0.50 0.70 0.50

0.0000001024192512

* Assuming we start in state 1, i.e., the first coin

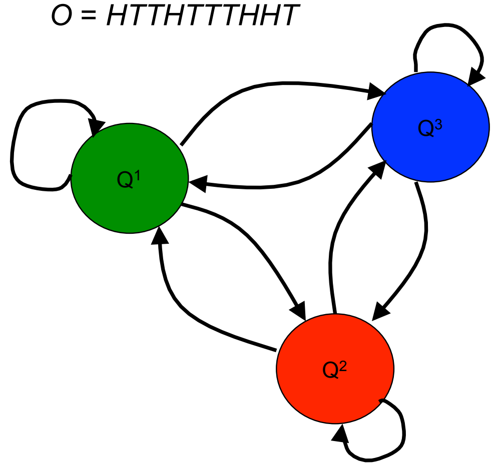
HMM: Model, λ

$$\lambda = (A, B)$$

Transition Probabilities, A	Emission Probabilities, B
$a_{11} = 0.90$	$b_1(H) = 0.9$
$a_{12} = 0.05$	$b_1(T) = 0.1$
$a_{13} = 0.05$	
$a_{21} = 0.02$	$b_2(H) = 0.5$
$a_{22} = 0.70$	$b_2(T) = 0.5$
$a_{23} = 0.28$	
$a_{31} = 0.30$	$b_3(H) = 0.2$
$a_{32} = 0.30$	$b_3(T) = 0.8$
$a_{33} = 0.40$	

Generating an Observation Sequence

$O = HTTHTTTHHT$

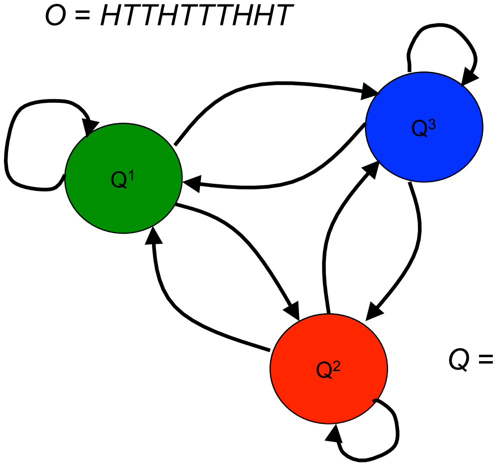


- Begin in the first state (i.e., the first coin)
- Emit an output character from the current state (i.e., flip the coin)
- Transition to the next state (i.e., choose a coin to flip next)
- Emit an output character from the current state (i.e., flip the coin)
- Transition to the next state (i.e., choose a coin to flip next)
- Emit an output character from the current state (i.e., flip the coin)

...

Hidden Information

$O = HTTHTTTHHT$



$Q = Q^1Q^1Q^3Q^3Q^3Q^3Q^3Q^1Q^1Q^1$

HMMs are Memoryless

The likelihood of a given future state depends only on the present state and not on past states

A Common Application of HMMs: Induction

Given an observation sequence $O = O_1O_2O_3 \ldots O_T$ and a model $\lambda = (A, B)$, what is the optimal state sequence?

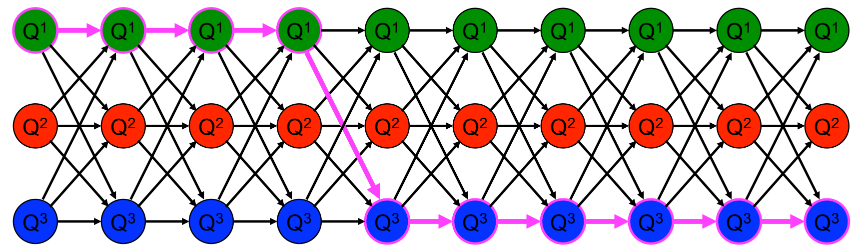
- We want to uncover the hidden part of the model. We want to maximize $P(Q|O, \lambda)$.

$$\arg\max_{Q_1, Q_2, \cdots, Q_T} \left(b_{Q_1}(O_1) a_{Q_1Q_2} b_{Q_2}(O_2) a_{Q_2Q_3} b_{Q_3}(O_3) \cdots a_{Q_{T-1}Q_T} b_{Q_T}(O_T) \right)$$

Path (State Sequence) Through HMM

Q^1 Q^1 Q^1 Q^1 Q^3 Q^3 Q^3 Q^3 Q^3 Q^3

$b_1(H)\ a_{11}\ b_1(T)\ a_{11}\ b_1(T)\ a_{11}\ b_1(H)\ a_{13}\ b_3(T)\ a_{33}\ b_3(T)\ a_{33}\ b_3(T)\ a_{33}\ b_3(H)\ a_{33}\ b_3(H)\ a_{33}\ b_3(T)$

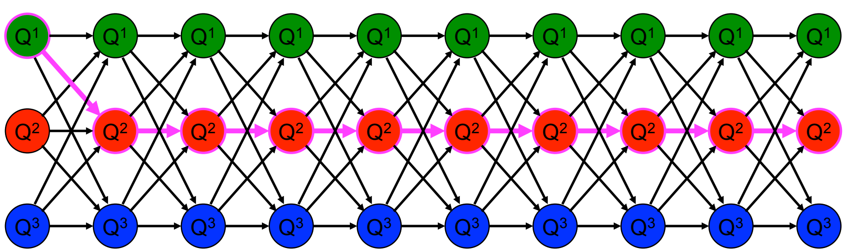


$t = 1$ 2 3 4 … T

Path (State Sequence) Through HMM

Q^1 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2

$b_1(H)\ a_{12}\ b_2(T)\ a_{22}\ b_2(T)\ a_{22}\ b_2(H)\ a_{22}\ b_2(T)\ a_{22}\ b_2(T)\ a_{22}\ b_2(T)\ a_{22}\ b_2(H)\ a_{22}\ b_2(H)\ a_{22}\ b_2(T)$

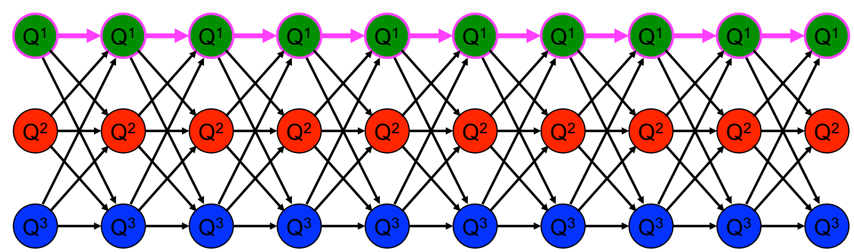


$t = 1$ 2 3 4 … T

Path (State Sequence) Through HMM

Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1 Q^1

$b_1(H)\ a_{11}\ b_1(T)\ a_{11}\ b_1(T)\ a_{11}\ b_1(H)\ a_{11}\ b_1(T)\ a_{11}\ b_1(T)\ a_{11}\ b_1(T)\ a_{11}\ b_1(H)\ a_{11}\ b_1(H)\ a_{11}\ b_1(T)$

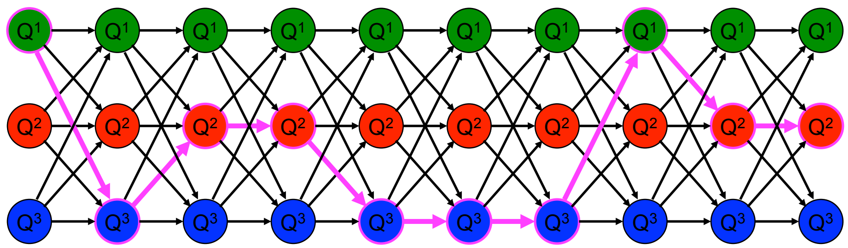


$t = 1$ 2 3 4 … T

Path (State Sequence) Through HMM

Q^1 Q^3 Q^2 Q^2 Q^3 Q^3 Q^3 Q^1 Q^2 Q^2

$b_1(H)\ a_{13}\ b_3(T)\ a_{32}\ b_2(T)\ a_{22}\ b_2(H)\ a_{23}\ b_3(T)\ a_{33}\ b_3(T)\ a_{33}\ b_3(T)\ a_{31}\ b_1(H)\ a_{12}\ b_2(H)\ a_{22}\ b_2(T)$

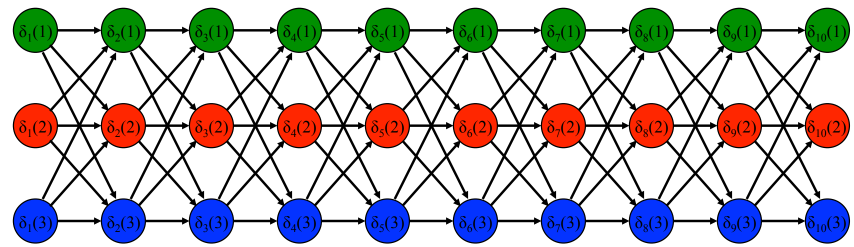


$t = 1$ 2 3 4 ... T

Viterbi Algorithm

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t = 1, j = 1 \\ 0.0 & \text{if } t = 1, j \neq 1 \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij}\right) * b_j(O_t) & \text{if } 2 \le t \le T \end{cases}$$

// base case, we start in state #1
// base case, we cannot start in
// a state other than state #1
// recursive case



$t = 1$ 2 3 4 ... T

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t = 1, j = 1 \\ 0.0 & \text{if } t = 1, j \neq 1 \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij}\right) * b_j(O_t) & \text{if } 2 \le t \le T \end{cases}$$

// base case, we start in state #1
// base case, we cannot start in
// a state other than state #1
// recursive case

$\delta_1(1)$	$\delta_2(1)$	$\delta_3(1)$	$\delta_4(1)$	$\delta_5(1)$	$\delta_6(1)$	$\delta_7(1)$	$\delta_8(1)$	$\delta_9(1)$	$\delta_{10}(1)$
$\delta_1(2)$	$\delta_2(2)$	$\delta_3(2)$	$\delta_4(2)$	$\delta_5(2)$	$\delta_6(2)$	$\delta_7(2)$	$\delta_8(2)$	$\delta_9(2)$	$\delta_{10}(2)$
$\delta_1(3)$	$\delta_2(3)$	$\delta_3(3)$	$\delta_4(3)$	$\delta_5(3)$	$\delta_6(3)$	$\delta_7(3)$	$\delta_8(3)$	$\delta_9(3)$	$\delta_{10}(3)$

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t = 1, j = 1 \\ 0.0 & \text{if } t = 1, j \neq 1 \end{cases}$$

// base case, we start in state #1
// base case, we cannot start in
// a state other than state #1

$b_1(O_1)$									
0.0									
0.0									

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} \\ \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij} \right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

$$\boldsymbol{\delta_7(2)} = \max\{\boldsymbol{\delta_6(1)} * a_{12}, \boldsymbol{\delta_6(2)} * a_{22}, \boldsymbol{\delta_6(3)} * a_{32}\} * b_2(O_7)$$

					$\delta_6(1)$				
					$\delta_6(2)$	$\delta_7(2)$			
					$\delta_6(3)$				

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} \\ \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij} \right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

$$\boldsymbol{\delta_4(3)} = \max\{\boldsymbol{\delta_3(1)} * a_{13}, \boldsymbol{\delta_3(2)} * a_{23}, \boldsymbol{\delta_3(3)} * a_{33}\} * b_3(O_4)$$

		$\delta_3(1)$							
		$\delta_3(2)$							
		$\delta_3(3)$	$\delta_4(3)$						

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} \\ \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij} \right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

$$\boldsymbol{\delta_9(1)} = \max\{\boldsymbol{\delta_8(1)} * a_{11}, \boldsymbol{\delta_8(2)} * a_{21}, \boldsymbol{\delta_8(3)} * a_{31}\} * b_1(O_9)$$

							$\delta_8(1)$	$\delta_9(1)$	
							$\delta_8(2)$		
							$\delta_8(3)$		

Probability of Optimal State Sequence?

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t = 1, j = 1 \quad \text{// base case, we start in state \#1} \\ 0.0 & \text{if } t = 1, j \ne 1 \quad \text{// base case, we cannot start in // a state other than state \#1} \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij} \right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

$\delta_1(1)$	$\delta_2(1)$	$\delta_3(1)$	$\delta_4(1)$	$\delta_5(1)$	$\delta_6(1)$	$\delta_7(1)$	$\delta_8(1)$	$\delta_9(1)$	$\delta_{10}(1)$
$\delta_1(2)$	$\delta_2(2)$	$\delta_3(2)$	$\delta_4(2)$	$\delta_5(2)$	$\delta_6(2)$	$\delta_7(2)$	$\delta_8(2)$	$\delta_9(2)$	$\delta_{10}(2)$
$\delta_1(3)$	$\delta_2(3)$	$\delta_3(3)$	$\delta_4(3)$	$\delta_5(3)$	$\delta_6(3)$	$\delta_7(3)$	$\delta_8(3)$	$\delta_9(3)$	$\delta_{10}(3)$

Backtracking Table

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t=1, j=1 \quad \text{// base case, we start in state \#1} \\ 0.0 & \text{if } t=1, j \neq 1 \quad \text{// base case, we cannot start in // a state other than state \#1} \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij}\right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

-1	1	1	1	1	1	3	3	1	1
-1	1	2	2	2	2	2	2	2	2
-1	1	3	3	2	2	2	2	2	2

Determining Optimal State Sequence

$Q^1 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2 Q^2$

-1	1	1	1	1	1	3	3	1	1
-1	1	2	2	2	2	2	2	2	2
-1	1	3	3	2	2	2	2	2	2

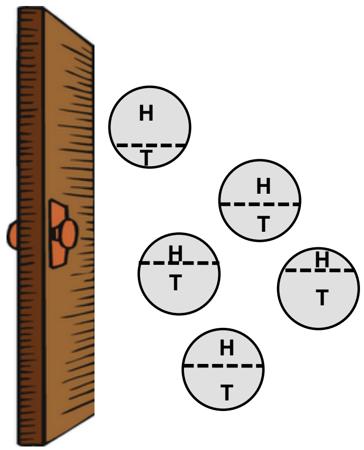
Runtime of Viterbi Algorithm?

$$\delta_t(j) = \begin{cases} b_1(O_1) & \text{if } t=1, j=1 \quad \text{// base case, we start in state \#1} \\ 0.0 & \text{if } t=1, j \neq 1 \quad \text{// base case, we cannot start in // a state other than state \#1} \\ \max\limits_{1 \le i \le N} \left(\delta_{t-1}(i) * a_{ij}\right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

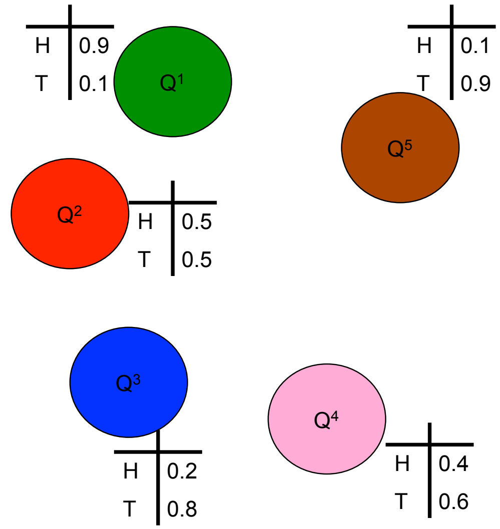
$\delta_1(1)$	$\delta_2(1)$	$\delta_3(1)$	$\delta_4(1)$	$\delta_5(1)$	$\delta_6(1)$	$\delta_7(1)$	$\delta_8(1)$	$\delta_9(1)$	$\delta_{10}(1)$
$\delta_1(2)$	$\delta_2(2)$	$\delta_3(2)$	$\delta_4(2)$	$\delta_5(2)$	$\delta_6(2)$	$\delta_7(2)$	$\delta_8(2)$	$\delta_9(2)$	$\delta_{10}(2)$
$\delta_1(3)$	$\delta_2(3)$	$\delta_3(3)$	$\delta_4(3)$	$\delta_5(3)$	$\delta_6(3)$	$\delta_7(3)$	$\delta_8(3)$	$\delta_9(3)$	$\delta_{10}(3)$

EXTENSION: Number of States

What If We Add More Coins?



HMM: Emission Probabilities, B



Emission Probabilities, B

$b_1(H) = 0.9$
$b_1(T) = 0.1$

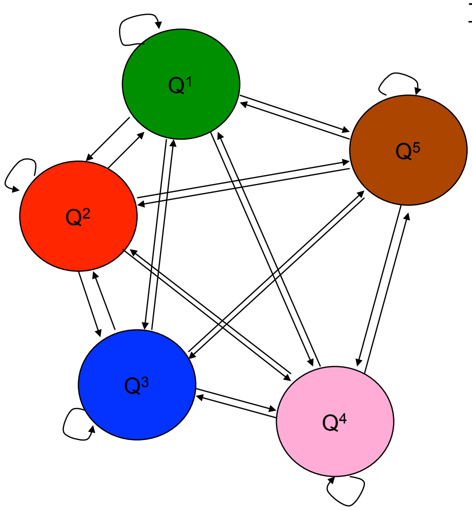
$b_2(H) = 0.5$
$b_2(T) = 0.5$

$b_3(H) = 0.2$
$b_3(T) = 0.8$

$b_4(H) = 0.4$
$b_4(T) = 0.6$

$b_5(H) = 0.1$
$b_5(T) = 0.9$

HMM: Transition Probability, A



Transition Probabilities, A

$a_{11} = 0.90$
$a_{12} = 0.02$
$a_{13} = 0.02$
$a_{14} = 0.03$
$a_{15} = 0.03$

$a_{21} = 0.10$
$a_{22} = 0.60$
$a_{23} = 0.15$
$a_{24} = 0.09$
$a_{25} = 0.06$

$a_{31} = 0.20$
$a_{32} = 0.15$
$a_{33} = 0.30$
$a_{34} = 0.25$
$a_{35} = 0.10$

$a_{41} = 0.04$
$a_{42} = 0.04$
$a_{43} = 0.04$
$a_{44} = 0.84$
$a_{45} = 0.04$

$a_{51} = 0.07$
$a_{52} = 0.22$
$a_{53} = 0.10$
$a_{54} = 0.11$
$a_{55} = 0.50$

HMM: Model, λ

$$\lambda = (A, B)$$

Emission Probabilities, B

$b_1(H) = 0.9$
$b_1(T) = 0.1$

$b_2(H) = 0.5$
$b_2(T) = 0.5$

$b_3(H) = 0.2$
$b_3(T) = 0.8$

$b_4(H) = 0.4$
$b_4(T) = 0.6$

$b_5(H) = 0.1$
$b_5(T) = 0.9$

Transition Probabilities, A

$a_{11} = 0.90$
$a_{12} = 0.02$
$a_{13} = 0.02$
$a_{14} = 0.03$
$a_{15} = 0.03$

$a_{21} = 0.10$
$a_{22} = 0.60$
$a_{23} = 0.15$
$a_{24} = 0.09$
$a_{25} = 0.06$

$a_{31} = 0.20$
$a_{32} = 0.15$
$a_{33} = 0.30$
$a_{34} = 0.25$
$a_{35} = 0.10$

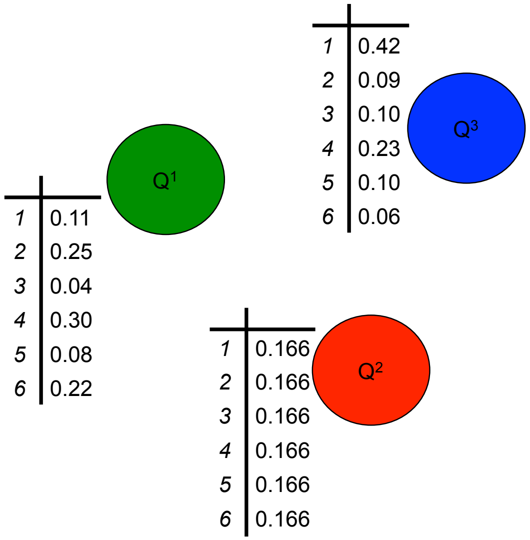
$a_{41} = 0.04$
$a_{42} = 0.04$
$a_{43} = 0.04$
$a_{44} = 0.84$
$a_{45} = 0.04$

$a_{51} = 0.07$
$a_{52} = 0.22$
$a_{53} = 0.10$
$a_{54} = 0.11$
$a_{55} = 0.50$

EXTENSION: Emission Probabilities

Instead of Coins…

HMM: Emission Probabilities, *B*



Emission Probabilities, *B*

$b_1(1) = 0.11$	$b_3(1) = 0.42$
$b_1(2) = 0.25$	$b_3(2) = 0.09$
$b_1(3) = 0.04$	$b_3(3) = 0.10$
$b_1(4) = 0.30$	$b_3(4) = 0.23$
$b_1(5) = 0.08$	$b_3(5) = 0.10$
$b_1(6) = 0.22$	$b_3(6) = 0.06$
$b_2(1) = 0.166$	
$b_2(2) = 0.166$	
$b_2(3) = 0.166$	
$b_2(4) = 0.166$	
$b_2(5) = 0.166$	
$b_2(6) = 0.166$	

HMM: Model, λ

$$\lambda = (A, B)$$

Emission Probabilities, *B*

$b_1(1) = 0.11$	$b_2(1) = 0.166$	$b_3(1) = 0.42$
$b_1(2) = 0.25$	$b_2(2) = 0.166$	$b_3(2) = 0.09$
$b_1(3) = 0.04$	$b_2(3) = 0.166$	$b_3(3) = 0.10$
$b_1(4) = 0.30$	$b_2(4) = 0.166$	$b_3(4) = 0.23$
$b_1(5) = 0.08$	$b_2(5) = 0.166$	$b_3(5) = 0.10$
$b_1(6) = 0.22$	$b_2(6) = 0.166$	$b_3(6) = 0.06$

Transition Probabilities, *A*

$a_{11} = 0.90$
$a_{12} = 0.05$
$a_{13} = 0.05$

$a_{21} = 0.02$
$a_{22} = 0.70$
$a_{23} = 0.28$

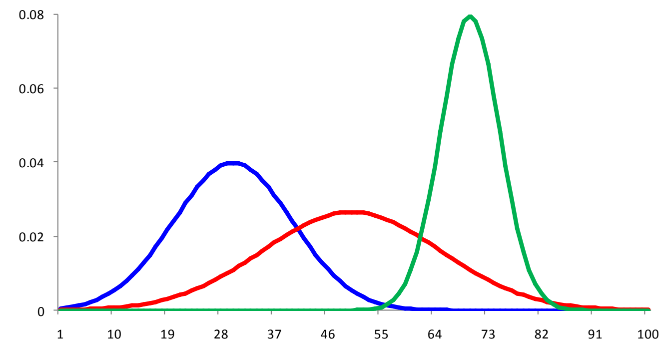
$a_{31} = 0.30$
$a_{32} = 0.30$
$a_{33} = 0.40$

EXTENSION: Continuous Observation Characters

Continuous Observation Characters

What if the alphabet of observation characters is not finite?

Suppose each observation character is a decimal number between 1 and 100.



HMM: Model, λ

$$\lambda = (A, B)$$

<u>Transition Probabilities, *A*</u>

$a_{11} = 0.90$

$a_{12} = 0.05$

$a_{13} = 0.05$

$a_{21} = 0.02$

$a_{22} = 0.70$

$a_{23} = 0.28$

$a_{31} = 0.30$

$a_{32} = 0.30$

$a_{33} = 0.40$

<u>Emission Probabilities, *B*</u>

$b_1(x)$ = Normal(x, 70, 5)

$b_2(x)$ = Normal(x, 50, 15)

$b_3(x)$ = Normal(x, 30, 10)

EXTENSION: Initial Probabilities

Initial Probabilities

We may not always want to start in the first state. Perhaps the first observation character was generated by a different state (other than the first).

We can have a probability of starting in each state:

$\pi_1 = 0.6$ $\pi_2 = 0.1$ $\pi_3 = 0.3$

HMM: Model, λ

$$\lambda = (A, B, \pi)$$

Transition Probabilities, A

$a_{11} = 0.90$
$a_{12} = 0.05$
$a_{13} = 0.05$

$a_{21} = 0.02$
$a_{22} = 0.70$
$a_{23} = 0.28$

$a_{31} = 0.30$
$a_{32} = 0.30$
$a_{33} = 0.40$

Emission Probabilities, B

$b_1(H) = 0.9$
$b_1(T) = 0.1$

$b_2(H) = 0.5$
$b_2(T) = 0.5$

$b_3(H) = 0.2$
$b_3(T) = 0.8$

Initial Probabilities, π

$\pi_1 = 0.6$

$\pi_2 = 0.1$

$\pi_3 = 0.3$

Dynamic Programming Table

$$\delta_t(j) = \begin{cases} \pi_j b_j(O_1) & \text{if } t = 1 \quad \text{// base case} \\ \max_{1 \le i \le N}\left(\delta_{t-1}(i) * a_{ij}\right) * b_j(O_t) & \text{if } 2 \le t \le T \quad \text{// recursive case} \end{cases}$$

$\delta_1(1)$	$\delta_2(1)$	$\delta_3(1)$	$\delta_4(1)$	$\delta_5(1)$	$\delta_6(1)$	$\delta_7(1)$	$\delta_8(1)$	$\delta_9(1)$	$\delta_{10}(1)$
$\delta_1(2)$	$\delta_2(2)$	$\delta_3(2)$	$\delta_4(2)$	$\delta_5(2)$	$\delta_6(2)$	$\delta_7(2)$	$\delta_8(2)$	$\delta_9(2)$	$\delta_{10}(2)$
$\delta_1(3)$	$\delta_2(3)$	$\delta_3(3)$	$\delta_4(3)$	$\delta_5(3)$	$\delta_6(3)$	$\delta_7(3)$	$\delta_8(3)$	$\delta_9(3)$	$\delta_{10}(3)$