Logistic Regression

CS344
Deep Learning

WELLESLEY

Common Problems ML May Help Solve

Logistic regression

BINARY CLASSIFICATION	MULTICLASS CLASSIFICATION	REGRESSION
Predicting 2 categorical outcomes	Predicting >2 categorical outcomes	Predicting a continuous outcome
Email is spam or not	Song is pop, rap, or country	Stock price
Someone has a disease or not	Flower is daisy, rose, sunflower, or tulip	Hours of sleep per night

Linear Classifier

$d = 2$

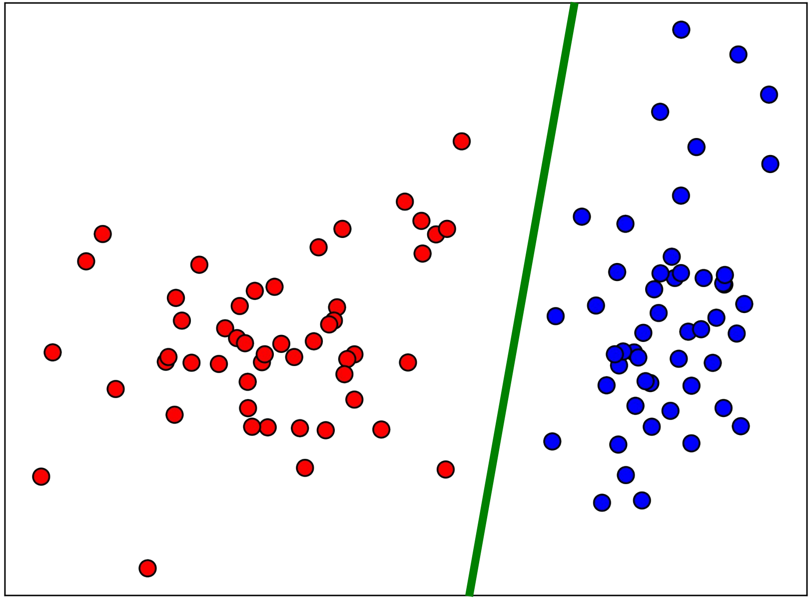
$m = 100$

35	59	0
72	63	1
13	77	0
22	33	0
58	19	1
…	…	…
80	37	1
44	51	0

Linear Classifier

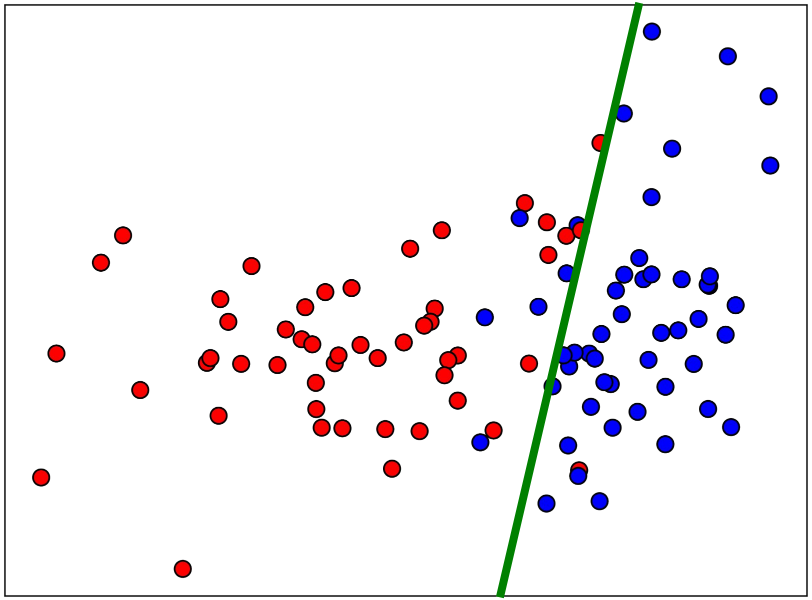
$$X = \begin{pmatrix} 35 & 59 \\ 72 & 63 \\ 13 & 77 \\ 22 & 33 \\ 58 & 19 \\ \dots & \dots \\ 80 & 37 \\ 44 & 51 \end{pmatrix} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \dots \\ 1 \\ 0 \end{bmatrix}$$

(100, 2) (100,)



Logistic regression learns a linear decision boundary, i.e., a *hyperplane* that divides the two classes

Linear Classifier

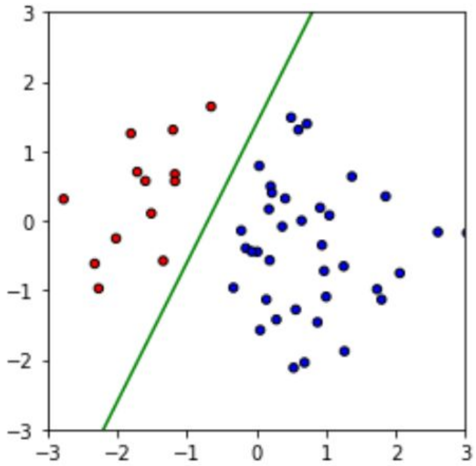


Data are not linearly separable

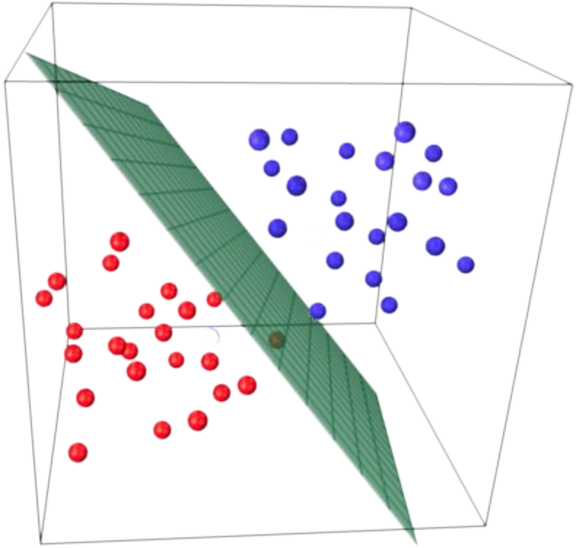
Hyperplane

A hyperplane in $\mathbb{R}^n$ is an n-1 dimensional subspace

A hyperplane in $\mathbb{R}^1$ is a point



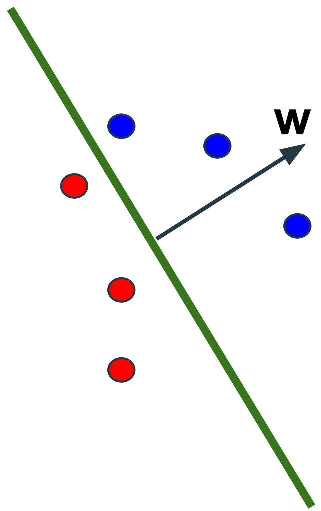
A hyperplane in $\mathbb{R}^2$ is a line



A hyperplane in $\mathbb{R}^3$ is a plane

What is a Hyperplane?

Parameterized by a “weight” vector **w** *orthogonal* to the hyperplane, centered at the origin

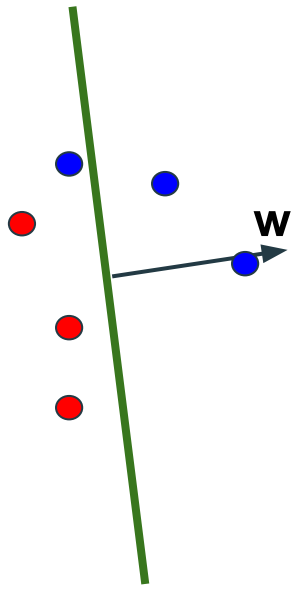


What range is

- the dot product of **w** with any of the blue points?
- the dot product of **w** with any of the red points?

What is a Hyperplane?

Parameterized by a “weight” vector **w** *orthogonal* to the hyperplane, centered at the origin



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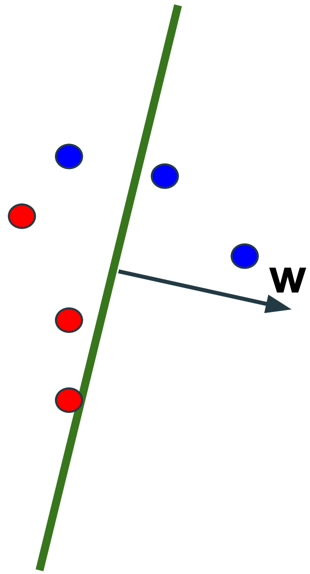

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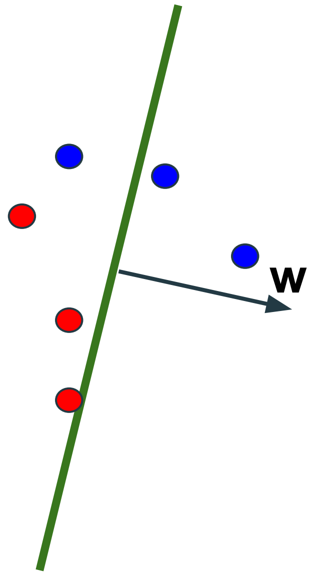
What is a Hyperplane?

Adding a bias term *b*

Parameterized by a "weight" vector **w** *orthogonal* to the hyperplane, centered at the origin



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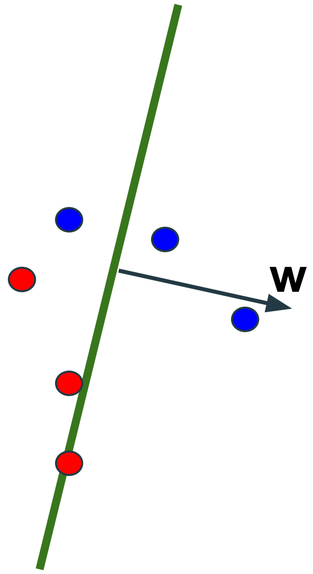

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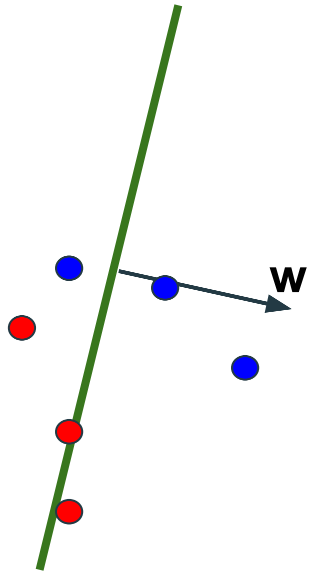


What range is

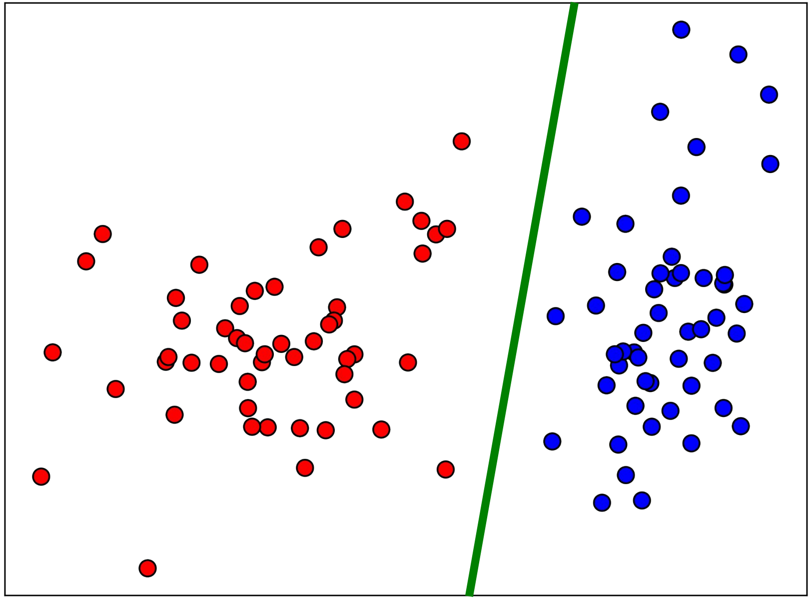
- the dot product of **w** with any of the blue points?
- the dot product of **w** with any of the red points?

Training

$b = 3$

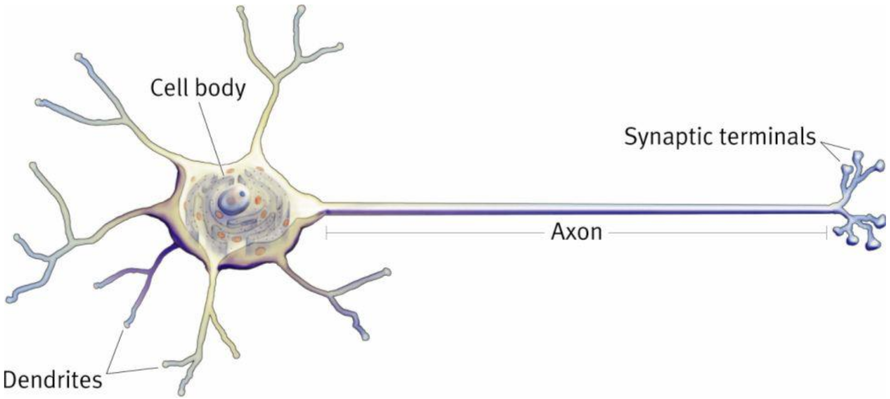
$w = (23 \quad 2)$

$$X = \begin{pmatrix} 35 & 59 \\ 72 & 63 \\ 13 & 77 \\ 22 & 33 \\ 58 & 19 \\ \dots & \dots \\ 80 & 37 \\ 44 & 51 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \dots \\ 1 \\ 0 \end{pmatrix}$$

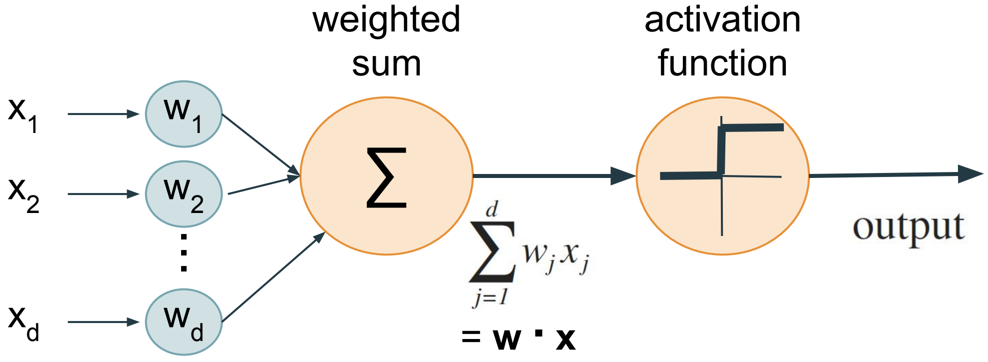


During *training*, the parameters of the model are learned from the *training data*

Neural Inspiration

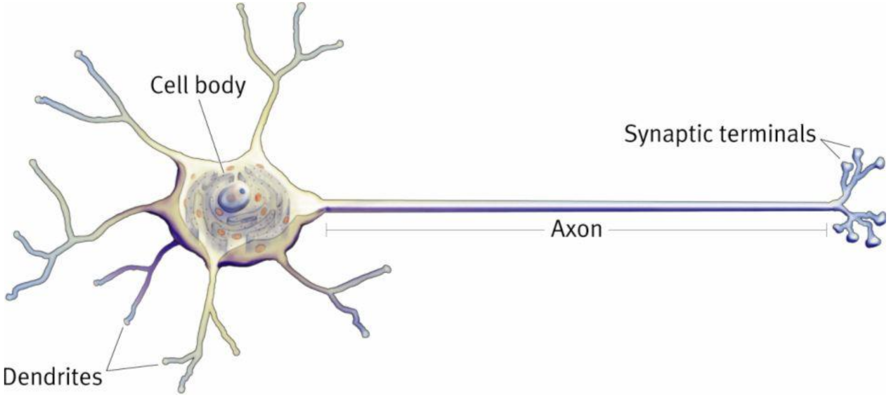


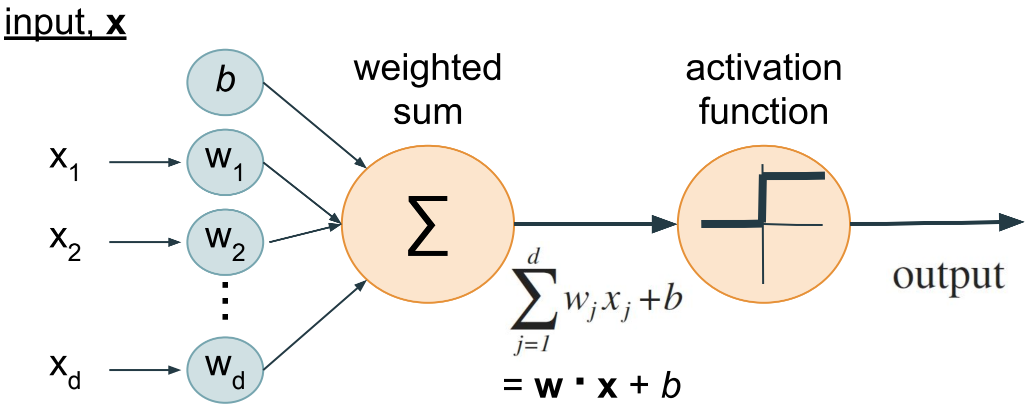
<u>input, **x**</u>



$$\text{output} = \begin{cases} 1 & \text{if } \mathbf{w} \cdot \mathbf{x} > 0 \\ 0 & \text{otherwise} \end{cases}$$

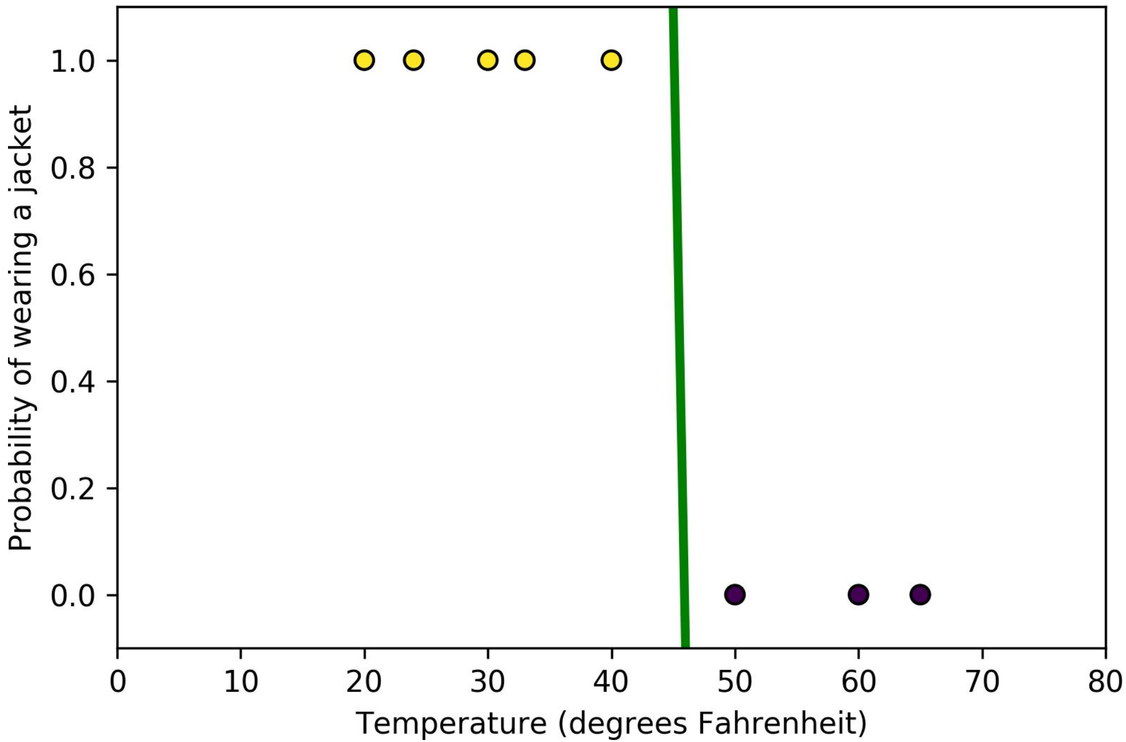
Neural Inspiration



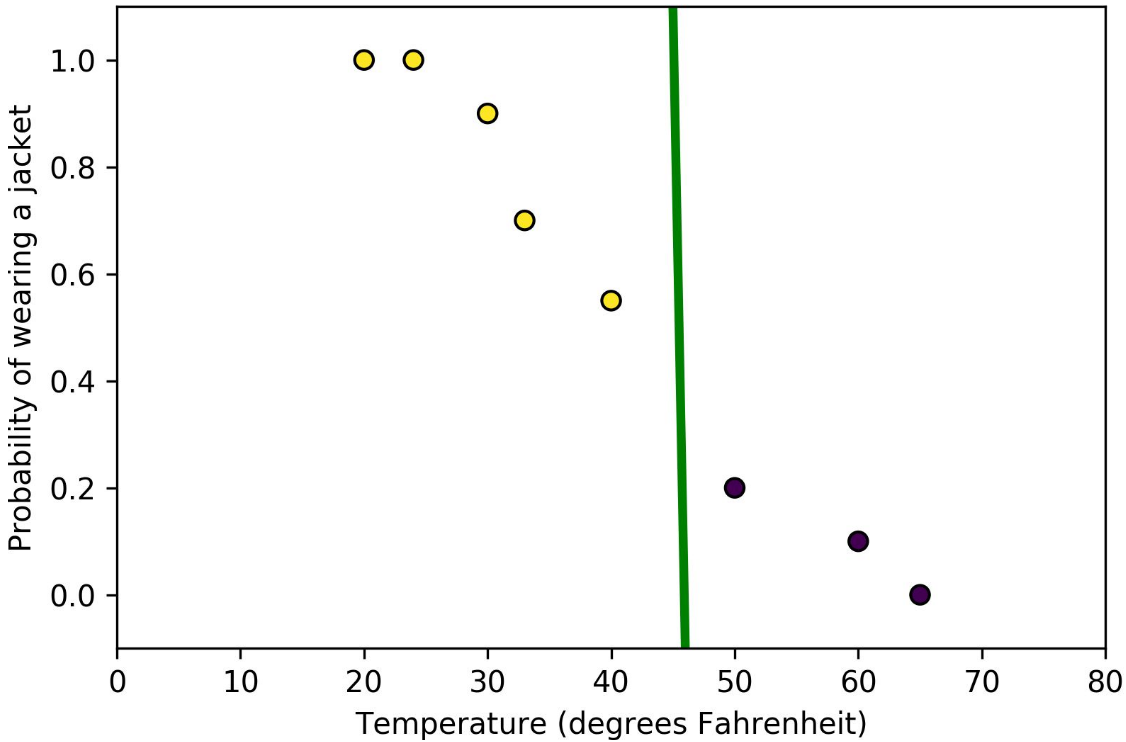


$$\text{output} = \begin{cases} 1 & \text{if } \mathbf{w} \cdot \mathbf{x} + b > 0 \\ 0 & \text{otherwise} \end{cases}$$

Should I wear a jacket?

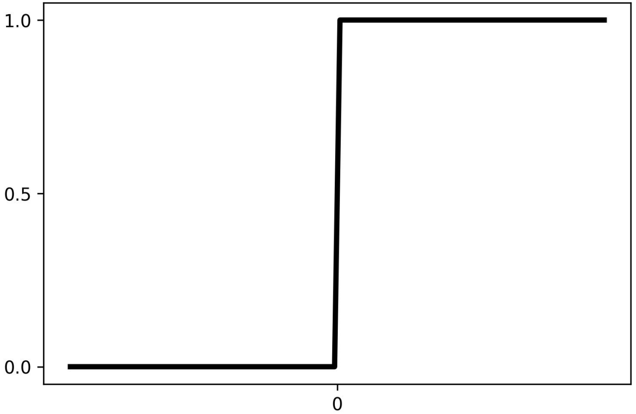


Should I wear a jacket?



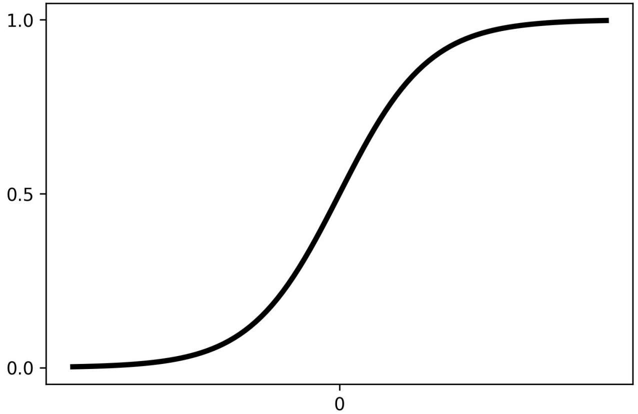
Hard Threshold vs. Sigmoid Function

Returns either 0 or 1



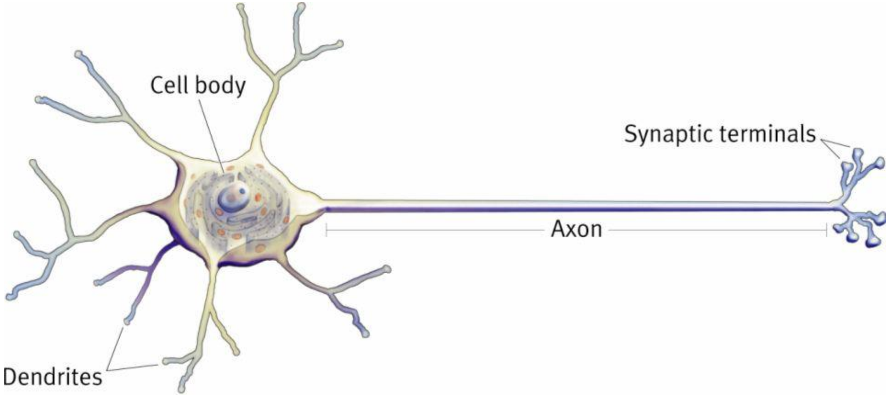
$$g(z) = \begin{cases} 0 \text{ if } & z<0 \\ 1 \text{ if } & z\geq 0 \end{cases}$$

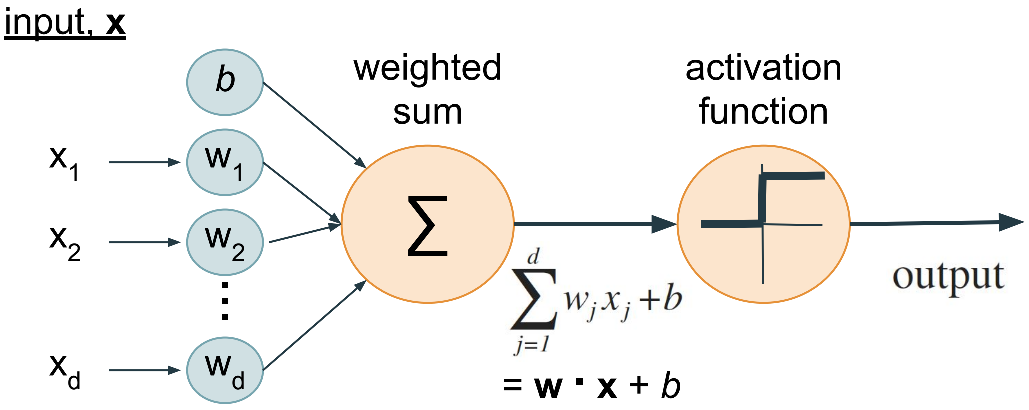
Returns a number between 0.0 and 1.0 that can be interpreted as a probability



$$g(z) = \frac{1}{1+e^{-z}}$$

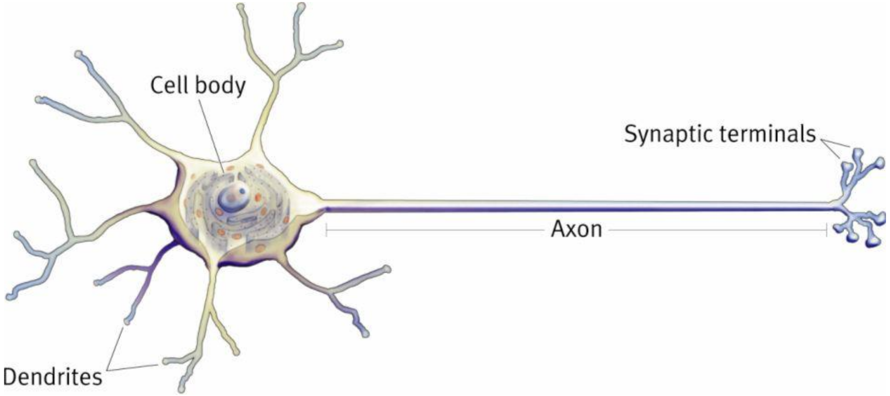
Neural Inspiration

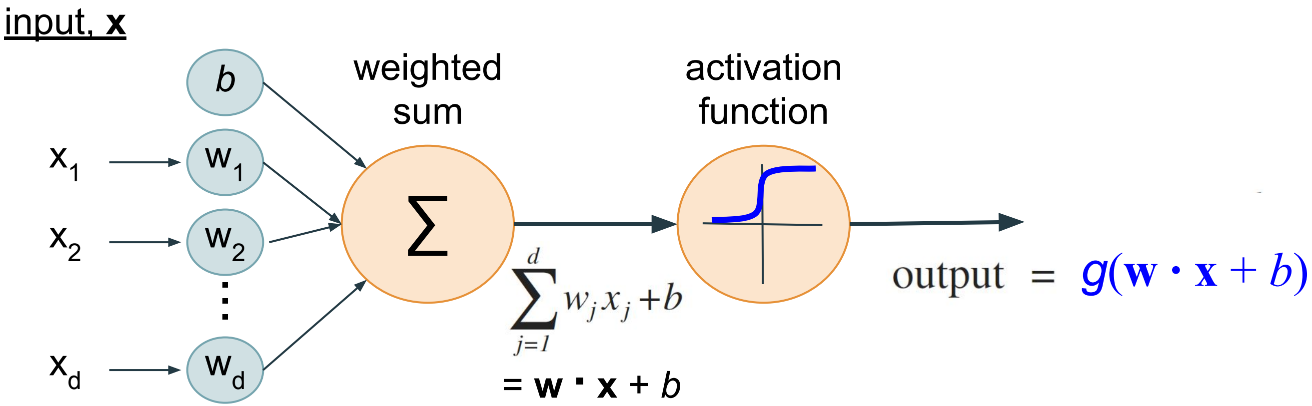




$$\text{output} = \begin{cases} 1 & \text{if } \mathbf{w} \cdot \mathbf{x} + b > 0 \\ 0 & \text{otherwise} \end{cases}$$

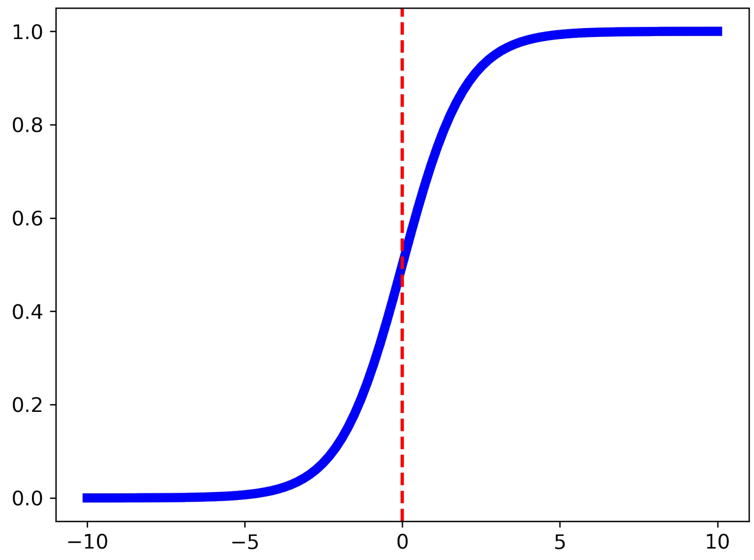
Neural Inspiration

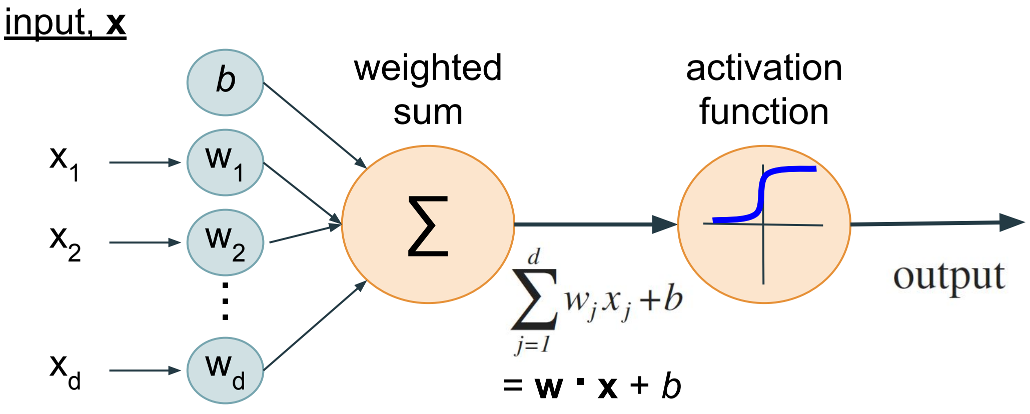




Sigmoid Function

$$g(z) = \frac{1}{1+e^{-z}}$$





$$\text{output} = g(\mathbf{w} \cdot \mathbf{x} + b)$$

$$= \frac{1}{1+e^{-(\mathbf{w}\cdot\mathbf{x}+b)}}$$

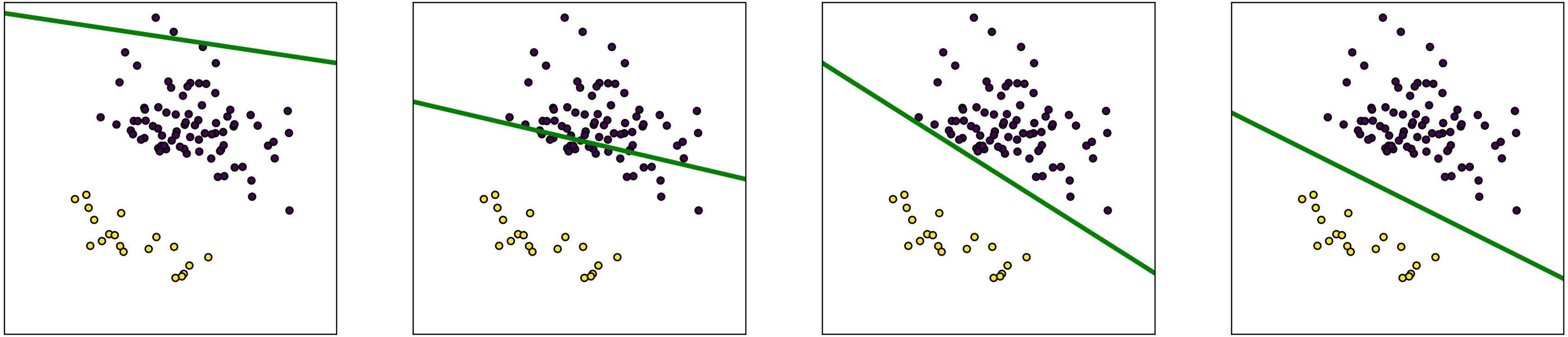
Forward Propagation

$$\hat{y} = g(\mathbf{w} \cdot \mathbf{x} + b) = \frac{1}{1 + e^{-(\mathbf{w} \cdot \mathbf{x} + b)}}$$

- $\hat{y}$ is interpreted as the probability that $y = 1$ for input $\mathbf{x}$
- For example, what is the probability that some email message $\mathbf{x}$ is spam (1) as opposed to ham (0)?
  - If $\hat{y}$ is 0.25, the probability that the message is spam is 25% and we classify the message as ham (0)
  - If $\hat{y}$ is 0.75, the probability that the message is spam is 75% and we classify the message as spam (1)

Parameters w and *b*

Different values for parameters **w** and *b* lead to different decision boundaries

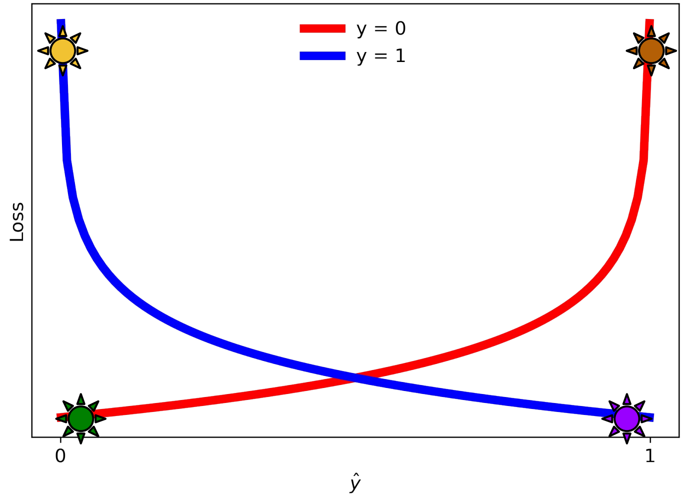


We want to quantify the *cost* associated with a given boundary (value settings for **w** and *b*) for our data

Then we can find the values of **w** and *b* that have the lowest *cost*

Loss

The *loss function*, L, quantifies the error, i.e., how far our prediction $\hat{y}$ is from the true label y



$$L = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$$

True label y	Prediction $\hat{y}$	Loss L
0	0.001	Close to 0
0	0.999	Large
1	0.999	Close to 0
1	0.001	Large

Cost

$$L = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$$

The *cost function*, J, is the average loss (error) of all *m* data points

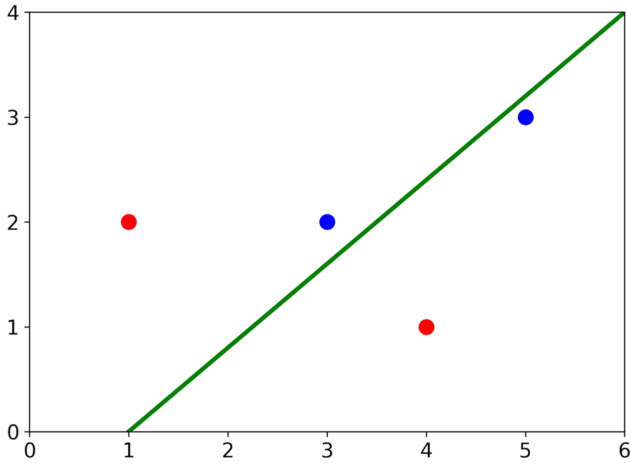
$$J = \frac{1}{m}\sum_{i=1}^{m} L$$

$$= \frac{1}{m}\sum_{i=1}^{m} -y_i\log(\hat{y}_i) - (1-y_i)\log(1-\hat{y}_i)$$

$$\hat{y} = g(\mathbf{w}\cdot\mathbf{x}+b) = \frac{1}{1+e^{-(\mathbf{w}\cdot\mathbf{x}+b)}}$$

Cost Examples

w = (4, -5)
b = -4



(4, 1)	(1, 2)	(3, 2)	(5, 3)	$\mathbf{x}$
7	-10	-2	1	$z = \mathbf{w} \cdot \mathbf{x} + b$
0.999	0.0004	0.12	0.73	$\hat{y} = g(z) = \dfrac{1}{1+e^{-z}}$
7.0009	0.0004	2.13	0.313	$L = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$

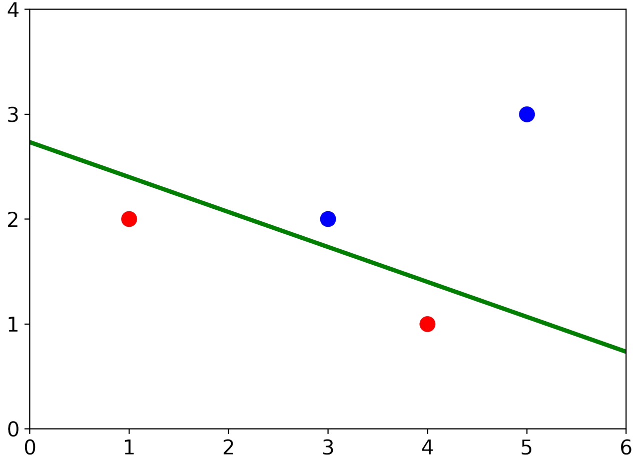
Average

2.36

$$J = \frac{1}{m}\sum_{i=1}^{m} L$$

Cost Examples

w = (1, 3)
b = -8.2



(4, 1)	(1, 2)	(3, 2)	(5, 3)	$\mathbf{x}$
-1.2	-1.2	-0.8	5.8	$z = \mathbf{w} \cdot \mathbf{x} + b$
0.23	0.23	0.69	0.997	$\hat{y} = g(z) = \dfrac{1}{1 + e^{-z}}$
0.26	0.26	0.37	0.003	$L = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$

Average

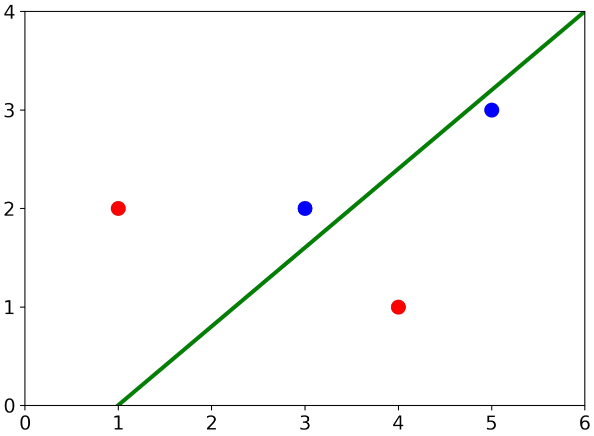
0.225

$$J = \frac{1}{m}\sum_{i=1}^{m} L$$

Cost Examples

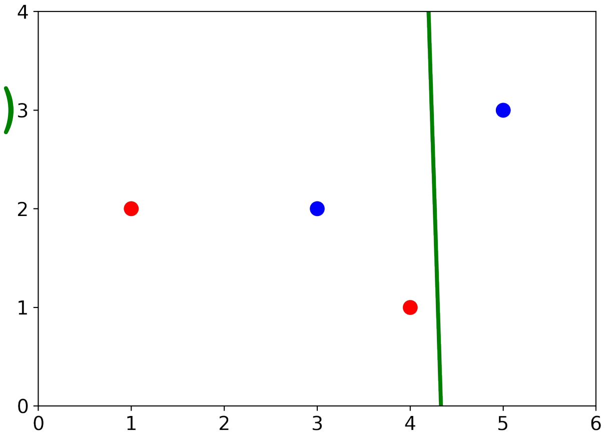
$w = (4, -5)$
$b = -4$

<u>Cost, J</u>
2.36



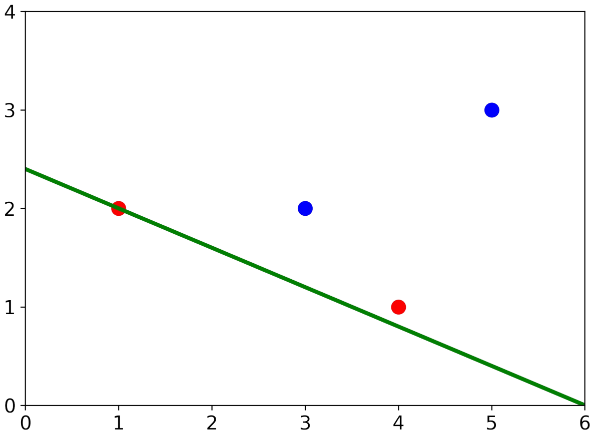
$w = (3, 0.1)$
$b = -13$

<u>Cost, J</u>
1.06



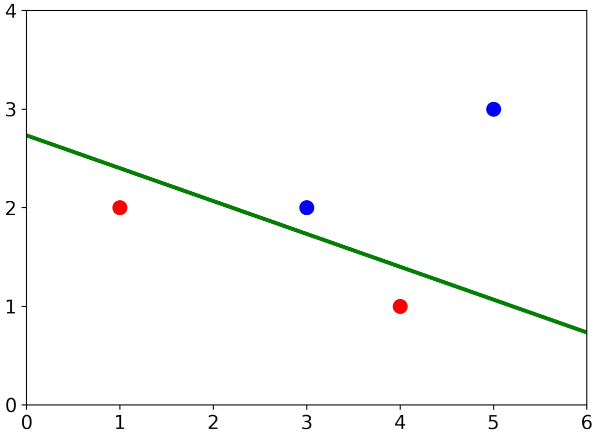
$w = (2, 5)$
$b = -12$

<u>Cost, J</u>
0.506

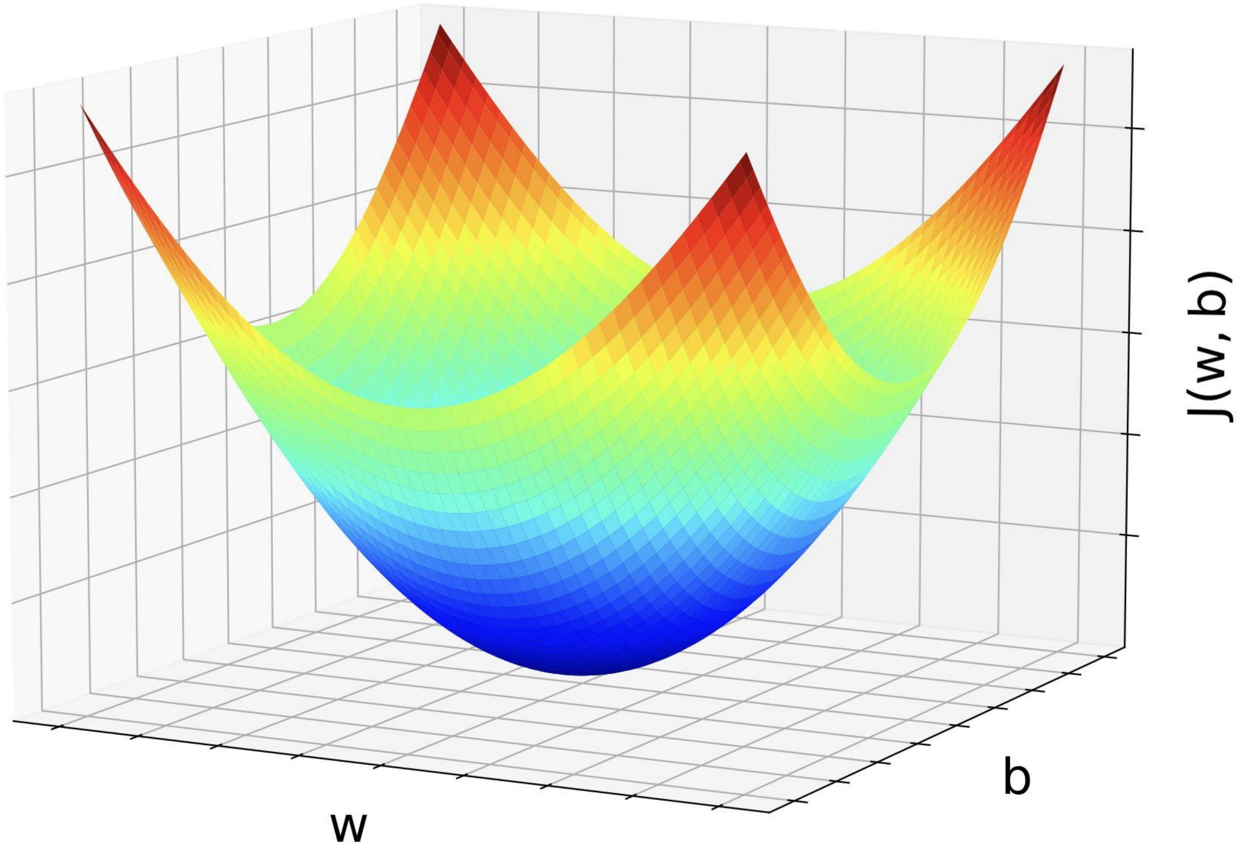


$w = (1, 3)$
$b = -8.2$

<u>Cost, J</u>
0.225



Cost Function



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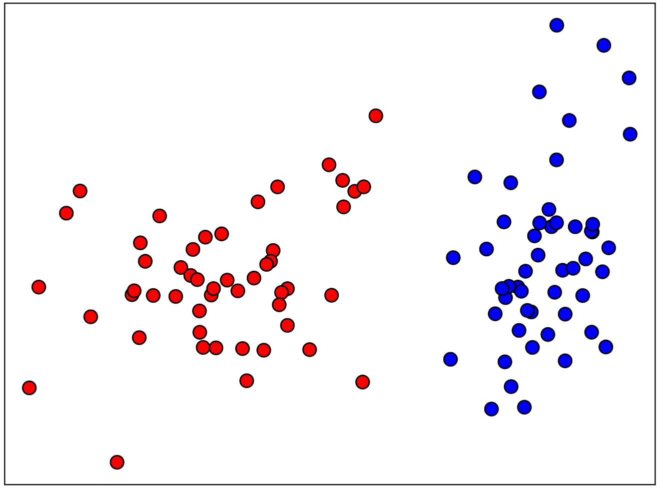
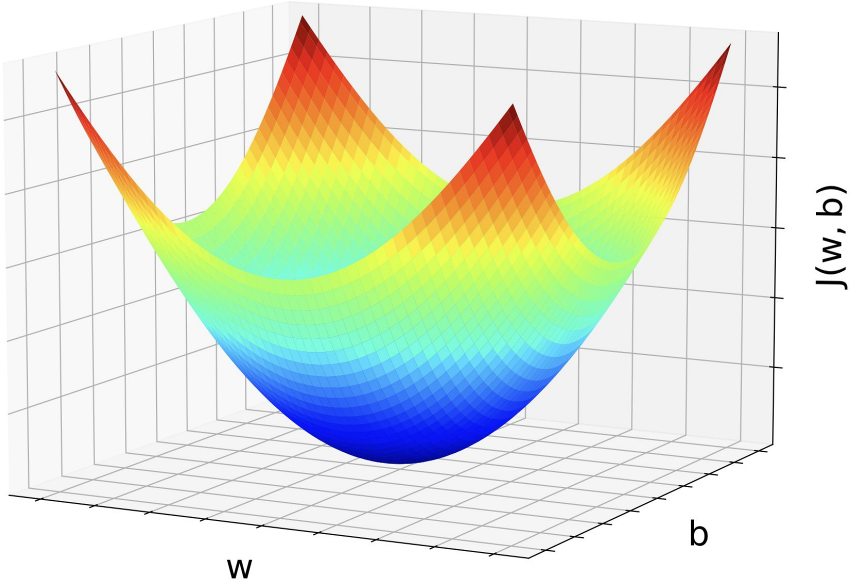
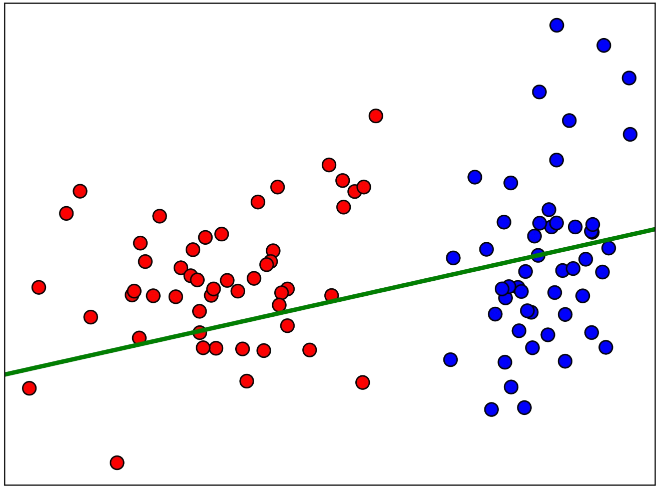


Data and Decision Boundary



Cost Function

Cost Function



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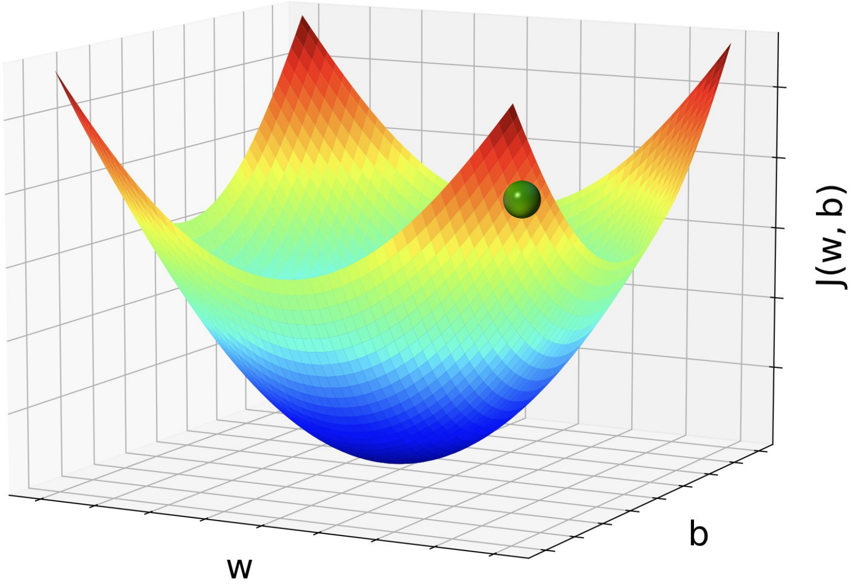
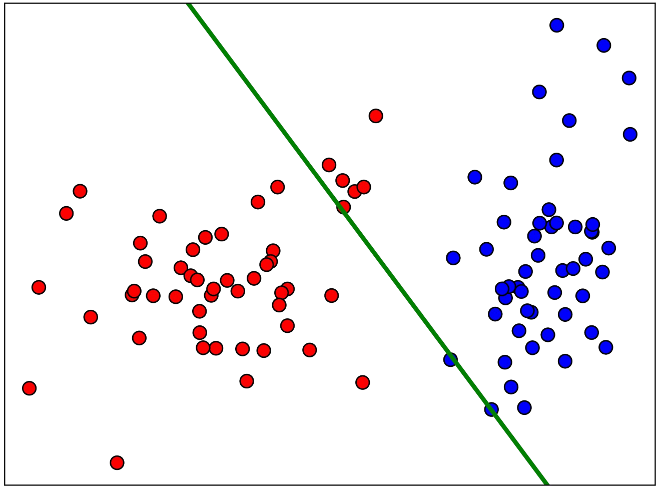


Cost Function

Cost Function



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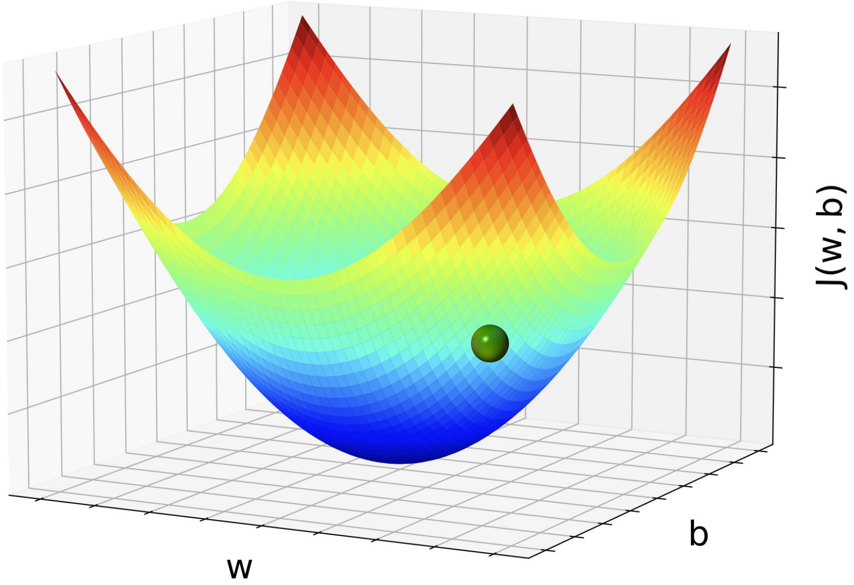
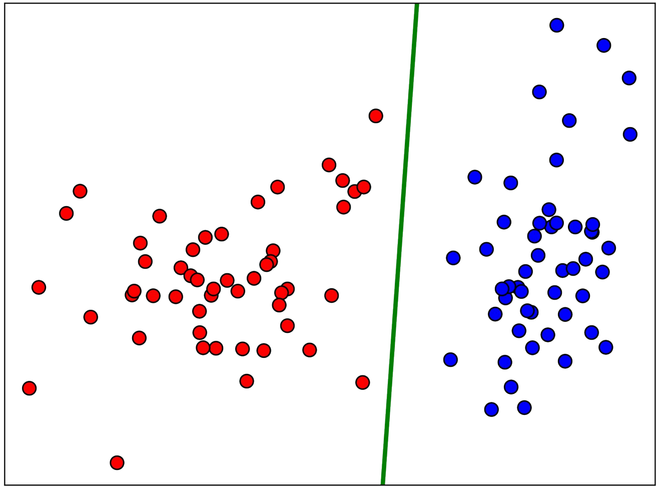


Cost Function

Cost Function



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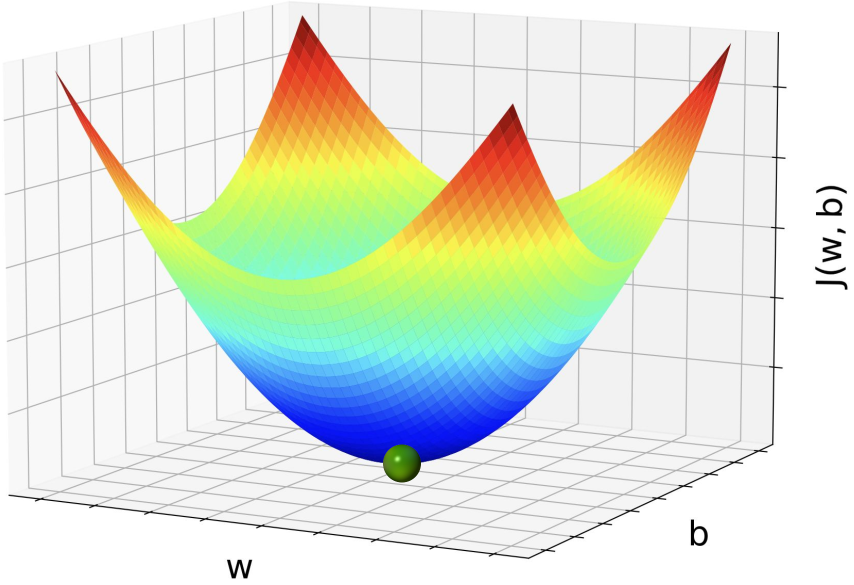

Cost Function