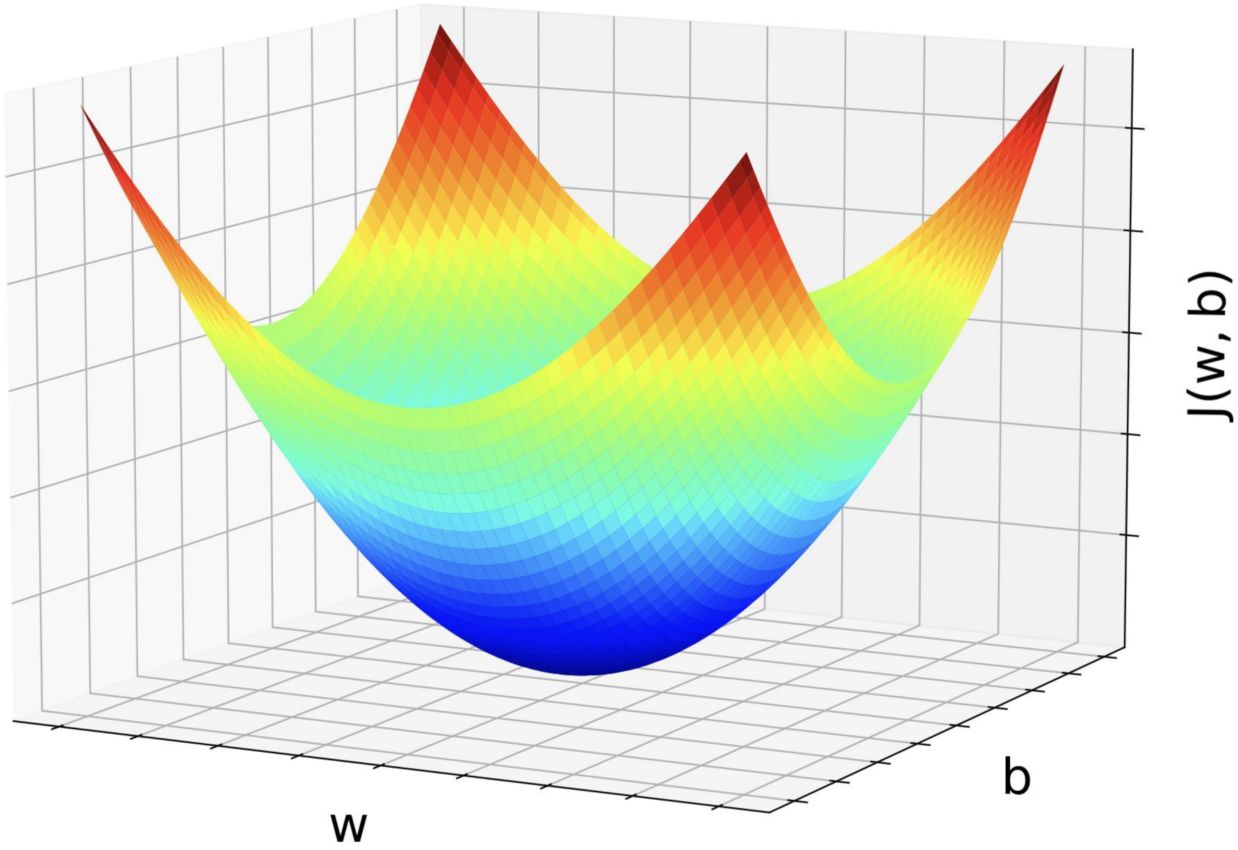
Gradient Descent

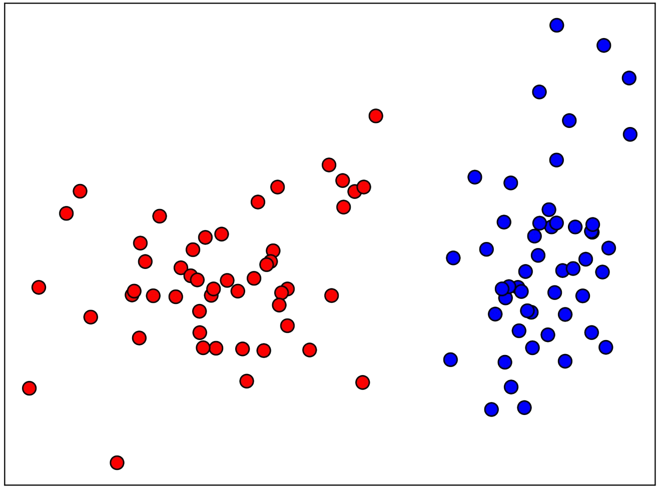
CS344
Deep Learning

WELLESLEY

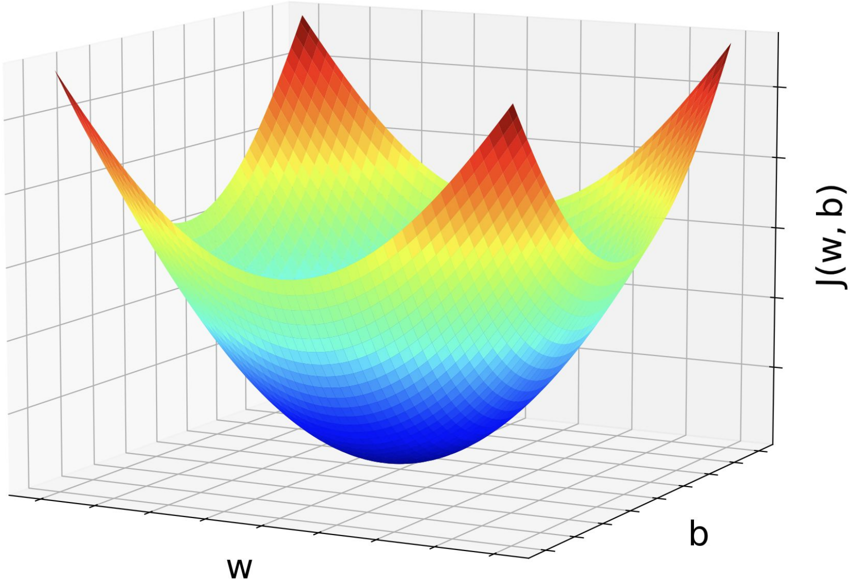
Cost Function



Cost Function



Data and Decision Boundary



Cost Function

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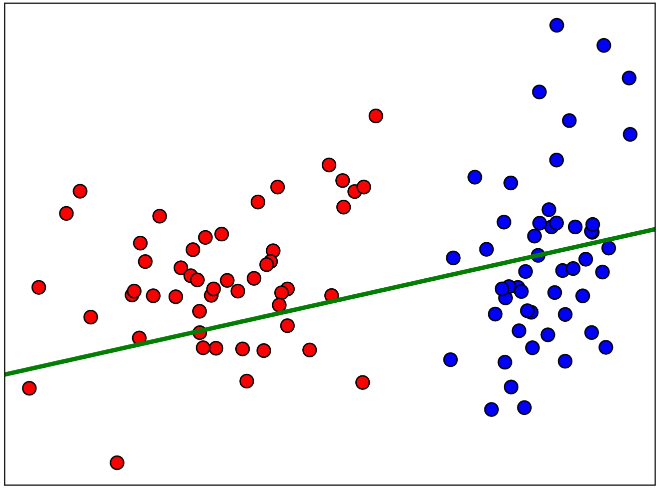
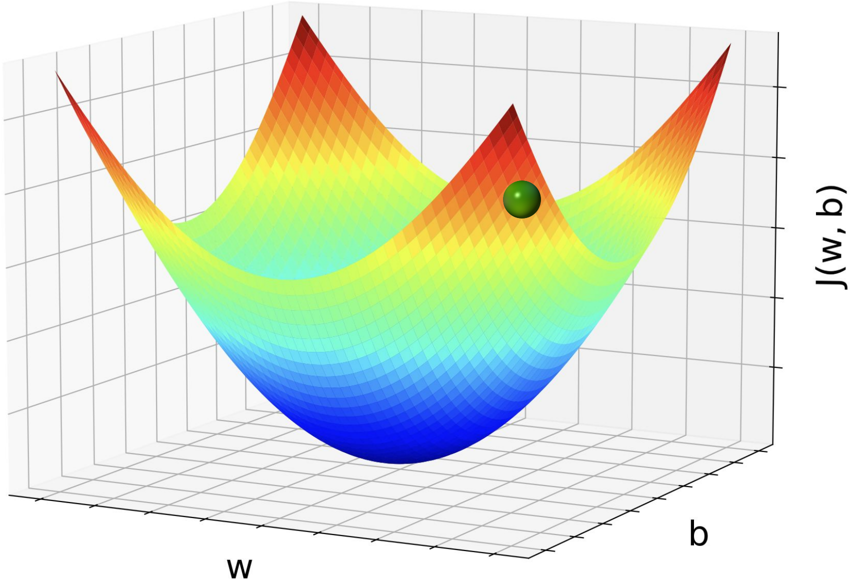


Data and Decision Boundary



Cost Function

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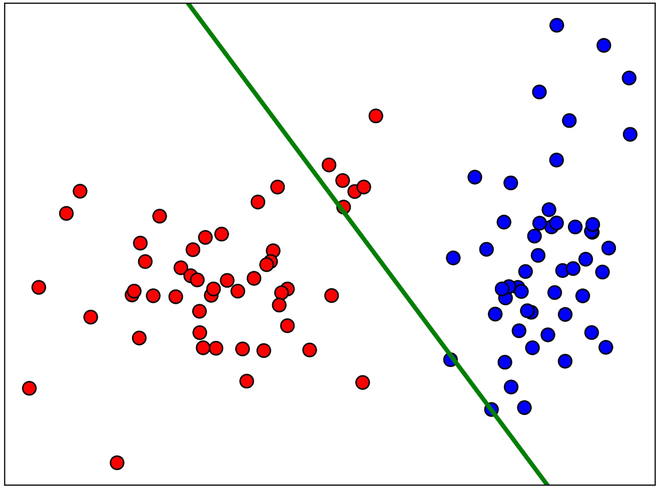

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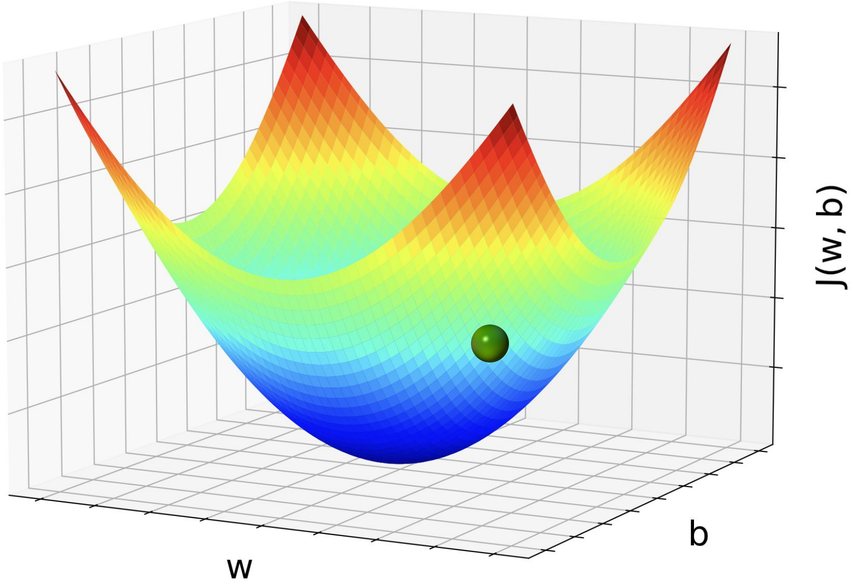

Cost Function

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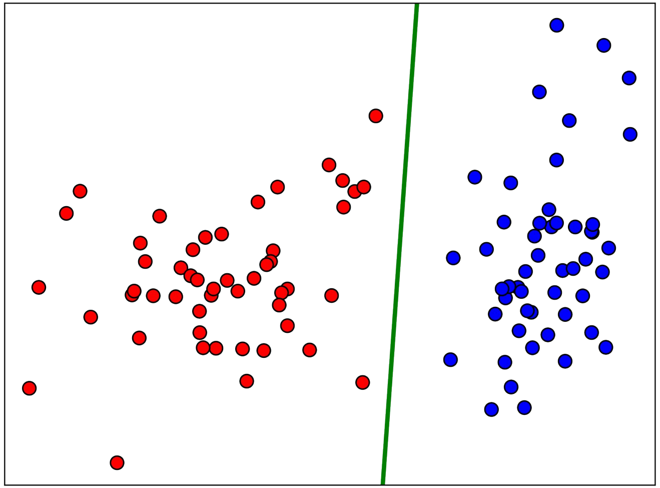
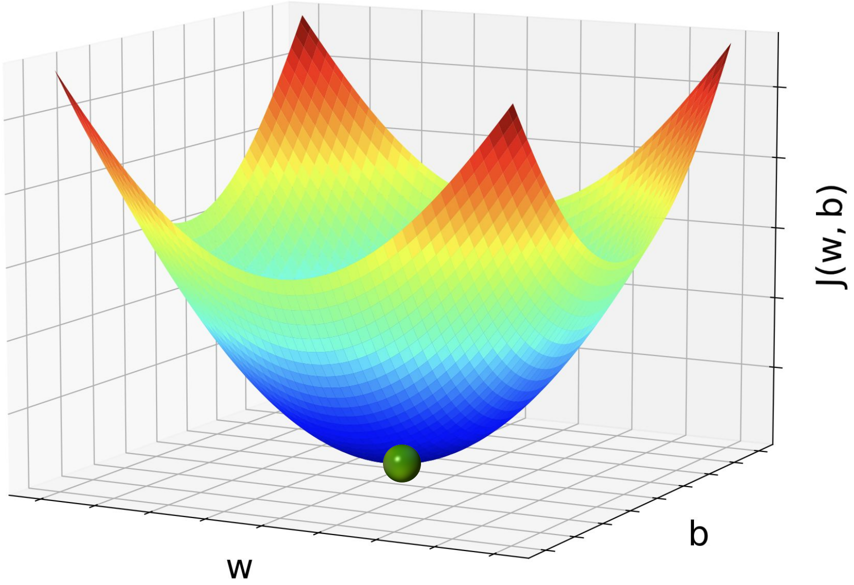


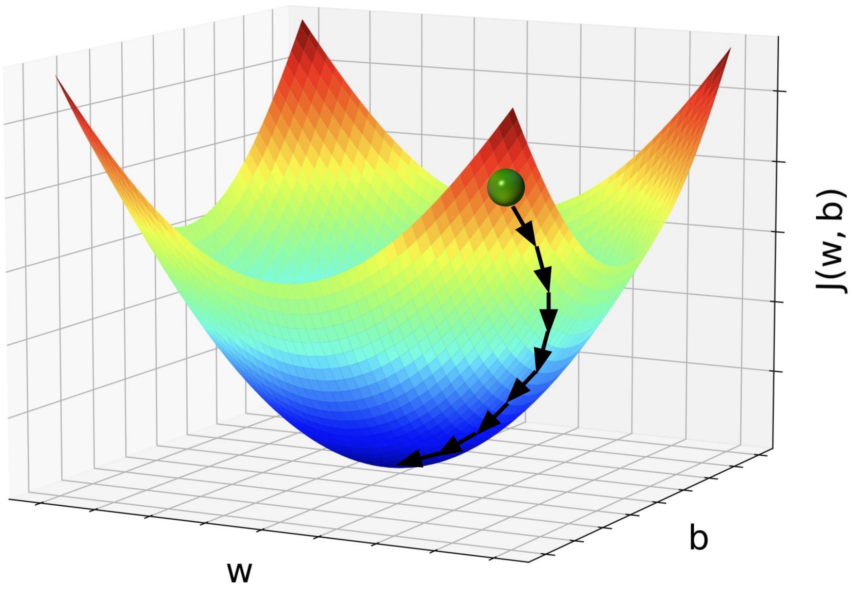
Data and Decision Boundary



Cost Function

Gradient Descent

We want to find parameters ***w*** and *b* that minimize the cost, J(***w***, *b*)



<u>Gradient Descent Algorithm</u>

- ❖ Initialize ***w*** and *b* (e.g., to 0)
- ❖ Repeat until converge:
  - ➢ Update ***w*** and *b* to reduce the cost J(***w***, *b*)

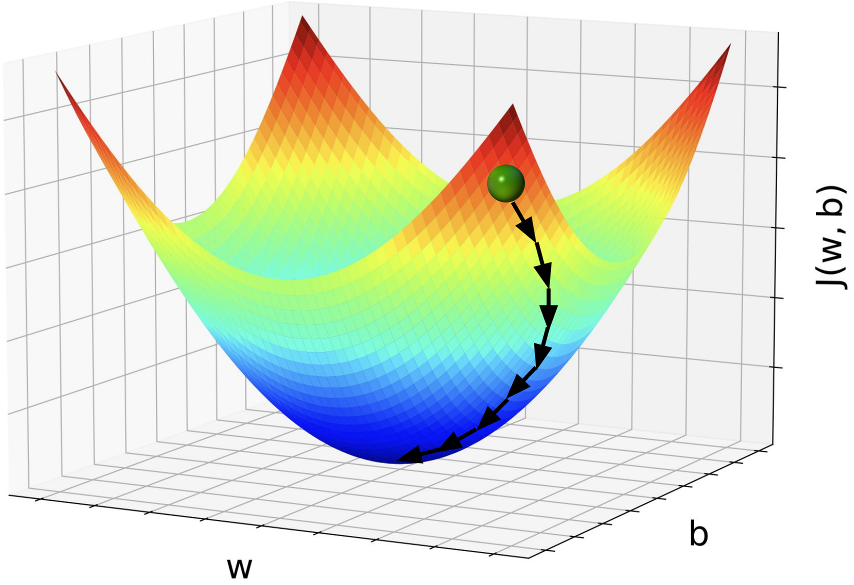
Gradient Descent Algorithm

Repeat until converge:

$$\mathbf{w} = \mathbf{w} - \alpha \frac{\partial}{\partial \mathbf{w}} J(\mathbf{w}, b)$$

$$b = b - \alpha \frac{\partial}{\partial b} J(\mathbf{w}, b)$$

α is the ***step size*** or ***learning rate***



Gradient Descent Algorithm

Repeat until converge:

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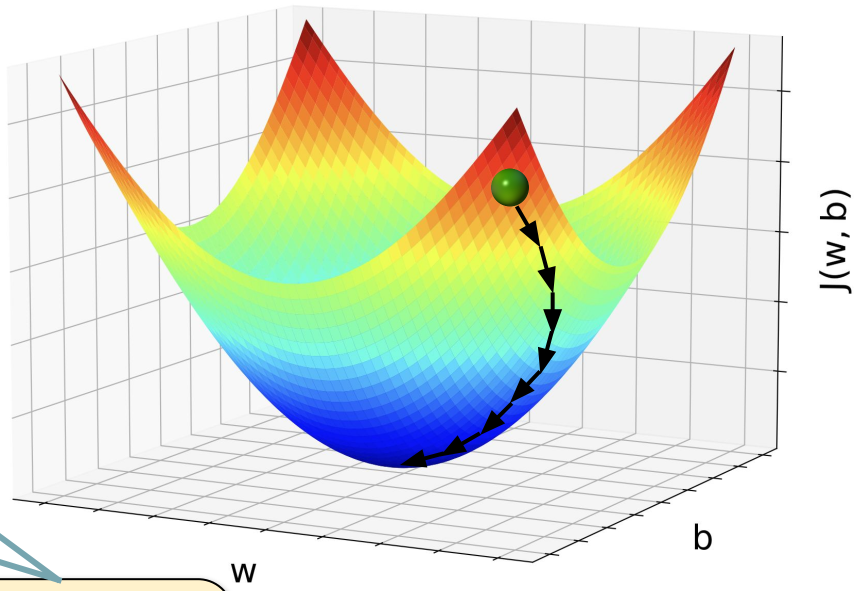
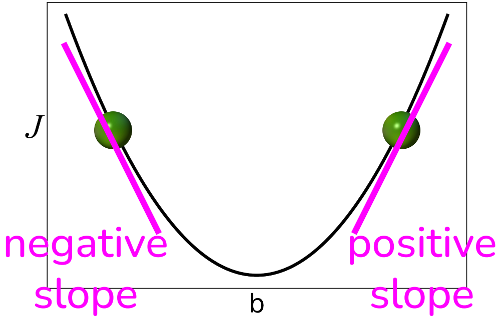




The partial derivative ∂ indicates the ***slope***, i.e., the ***direction*** to step

Computing Derivatives: Chain Rule

$f(x) = x^2$

$f'(x) = 2x$

$g(x) = 2x+1$

$g'(x) = 2$

$h(x) = f(g(x)) = (2x+1)^2$

$h'(x) = 2 \cdot (2x+1) \cdot 2$

$h'(x) = f'(g(x)) \cdot g'(x)$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Computing Derivatives: Chain Rule

$$z = \mathbf{w} \cdot \mathbf{x} + b$$

$$\hat{y} = g(z) = \frac{1}{1 + e^{-z}}$$

$$L = -y\log(\hat{y}) - (1 - y)\log(1 - \hat{y})$$

$$J = \frac{1}{m} \sum_{i=1}^{m} L$$

Computing Derivatives: Chain Rule

$$z = \mathbf{w} \cdot \mathbf{x} + b$$

$$\mathbf{x} \qquad \frac{dz}{dw}$$

$$a = g(z) = \frac{1}{1 + e^{-z}}$$

$$a(1 - a) \qquad \frac{da}{dz}$$

$$L = -y\log(a) - (1 - y)\log(1 - a)$$

$$\frac{-y}{a} + \frac{1 - y}{1 - a} \qquad \frac{dL}{da}$$

$$J = \frac{1}{m}\sum_{i=1}^{m} L$$

$$\frac{dL}{dw} = \frac{dL}{da} \cdot \frac{da}{dz} \cdot \frac{dz}{dw} = (a - y) \cdot x$$

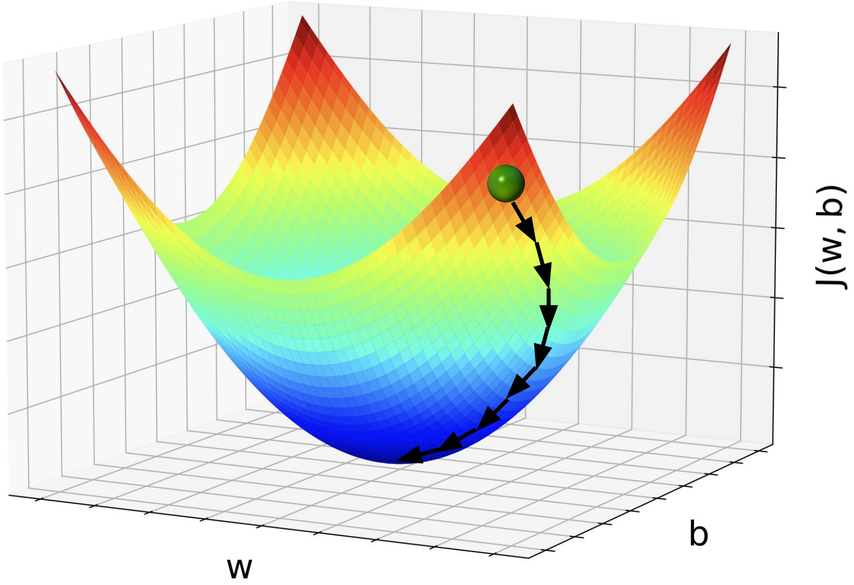
$$\frac{dJ}{dw} = \frac{1}{m}\sum_{i=1}^{m} \frac{dL}{dw} = \frac{1}{m}\sum_{i=1}^{m}(a - y) \cdot x$$

Gradient Descent Algorithm

Repeat until converge:

$$\mathbf{w} = \mathbf{w} - \alpha \boxed{\frac{\partial}{\partial \mathbf{w}} J(\mathbf{w}, b)}$$

$$b = b - \alpha \boxed{\frac{\partial}{\partial b} J(\mathbf{w}, b)}$$



Gradient Descent Algorithm

Repeat until converge:

$$\mathbf{w} = \mathbf{w} - \alpha \boxed{\frac{1}{m}\sum_{i=1}^{m}}(a - y) \cdot x$$

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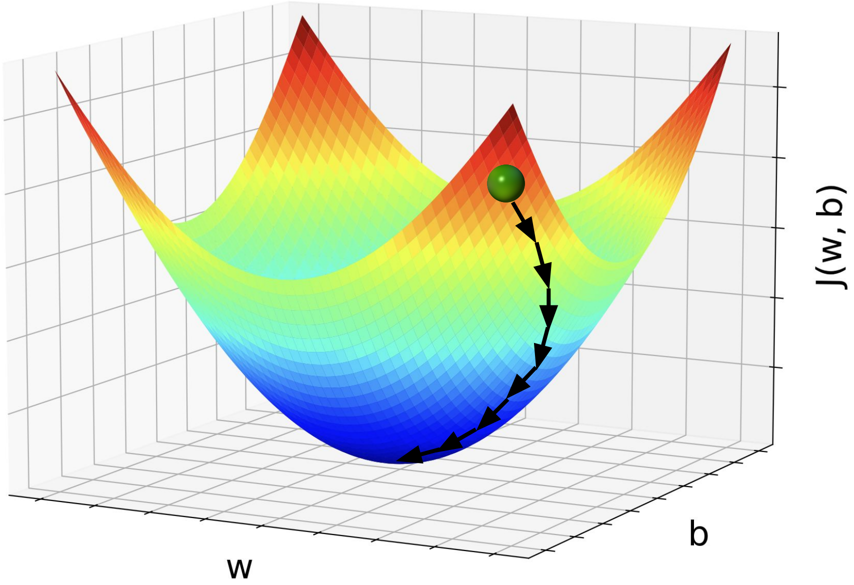


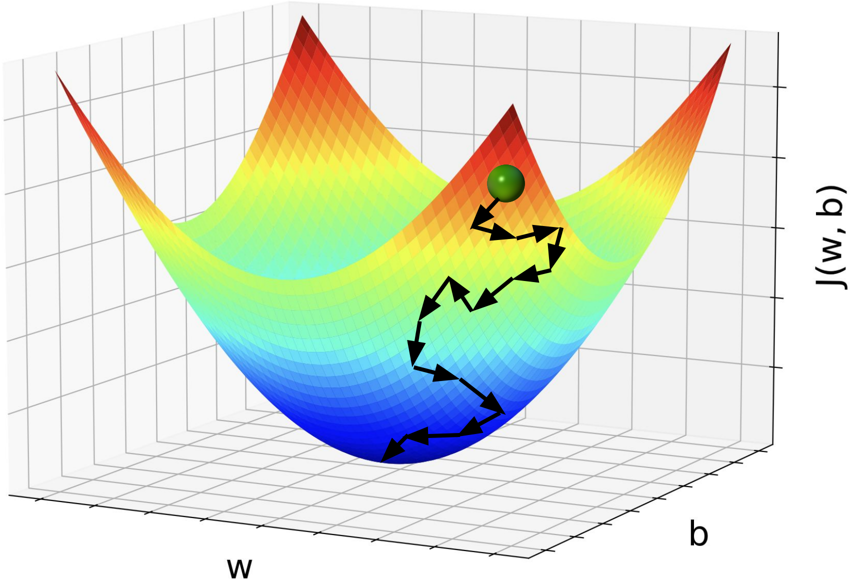
With ***batch*** gradient descent,
we consider ***all*** data points each
time we update a weight parameter

Gradient Descent Algorithm

Repeat until converge:

$$\mathbf{w} = \mathbf{w} - \alpha\ (a - y) \cdot x$$

$$b = b - \alpha\ (a - y)$$



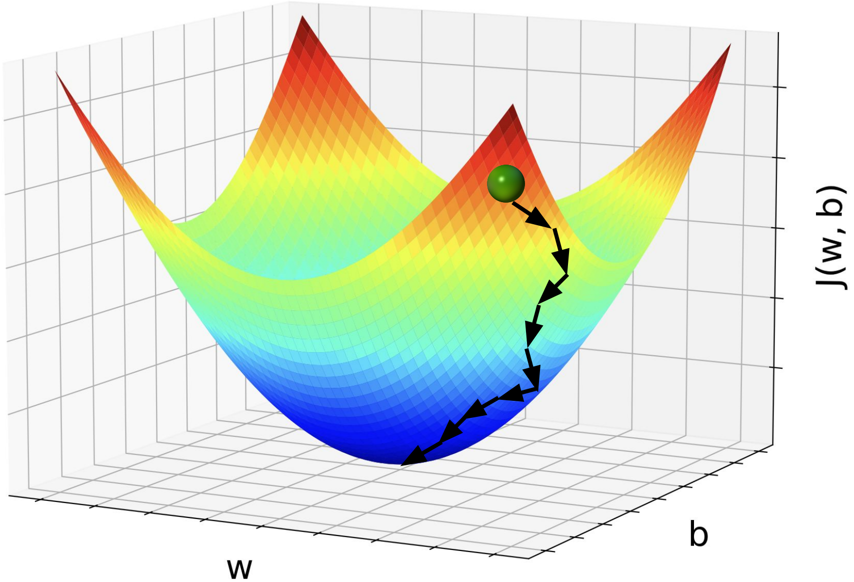
With ***stochastic*** gradient descent, we consider a ***single*** data point each time we update a weight parameter

Gradient Descent Algorithm

Repeat until converge:

$$\mathbf{w} = \mathbf{w} - \alpha \frac{1}{batch\_size} \sum_{i=1}^{batch\_size} (a - y) \cdot x$$

$$b = b - \alpha \frac{1}{batch\_size} \sum_{i=1}^{batch\_size} (a - y)$$



With ***mini-batch*** gradient descent, we consider a ***small batch*** of data points each time we update a weight parameter

Vectorization

$$z = \mathbf{x} \cdot \mathbf{w} + b$$

```
Z = new array of shape (m, 1)
For i = 1 to m:
    Z[i] = np.dot(X[i], w) + b
```

$$X_{(m,2)} = \begin{bmatrix} 3 & 2 \\ 9 & 1 \\ 5 & 4 \\ 2 & 3 \\ 0 & 4 \\ \dots & \dots \\ 3 & 1 \end{bmatrix} \qquad \mathbf{w}_{(2,1)} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad b = 1 \qquad Z_{(m,1)} = \begin{bmatrix} 2 \\ -6 \\ 4 \\ 5 \\ 9 \\ \dots \\ 0 \end{bmatrix}$$

Vectorization speeds up execution time dramatically

Highly optimized libraries ***parallelize*** computation

Guideline: ***avoid looping*** over arrays

`numpy` has a built in function for everything

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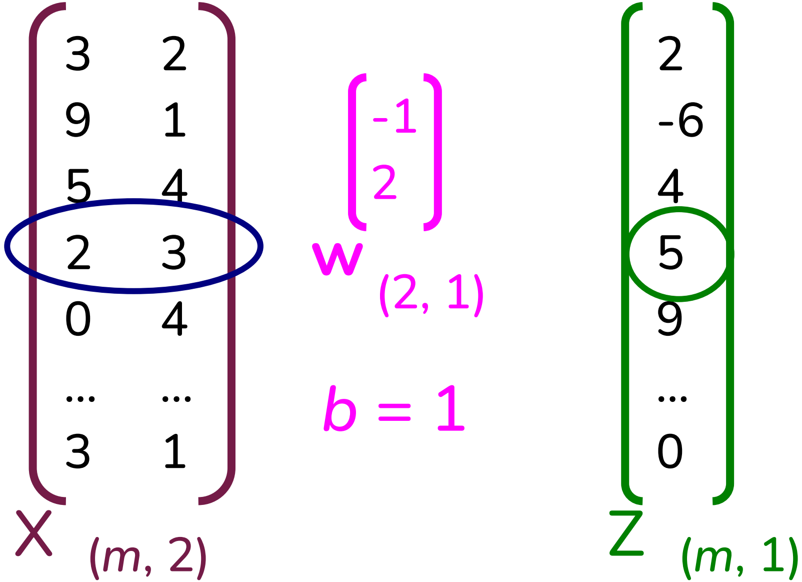

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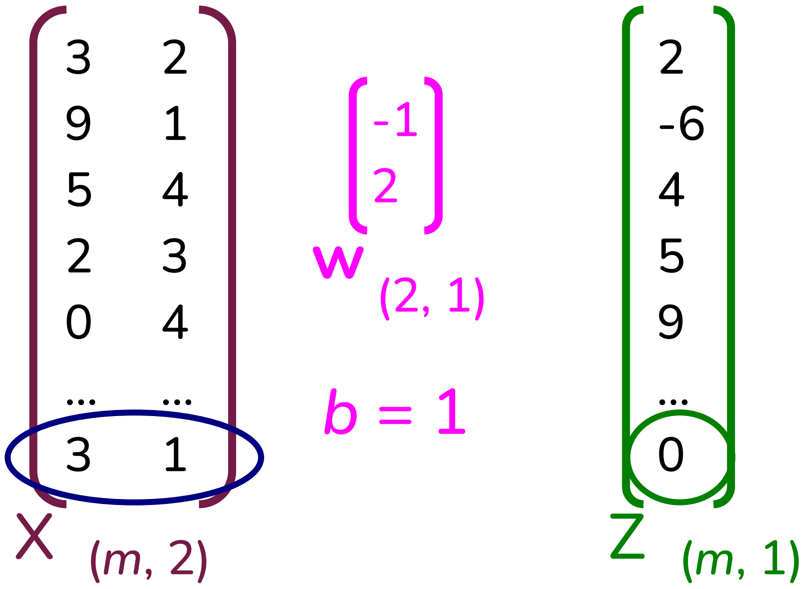

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Vectorization

$$a = \text{sigmoid}(z) = \frac{1}{1 + e^{-z}}$$

```
A = sigmoid(Z)
```

$$Z = \begin{bmatrix} 2 \\ -6 \\ 4 \\ 5 \\ 9 \\ \dots \\ 0 \end{bmatrix} \quad (m, 1) \qquad A = \begin{bmatrix} 0.881 \\ 0.002 \\ 0.982 \\ 0.993 \\ 0.999 \\ \dots \\ 0.5 \end{bmatrix} \quad (m, 1)$$

Vectorization speeds up execution time dramatically

Highly optimized libraries ***parallelize*** computation

Guideline: ***avoid looping*** over arrays

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Training (Fitting)

<u>Gradient Descent Algorithm</u>

- ❖ Initialization
- ❖ Repeat until convergence:
  - ➢ Forward propagation
  - ➢ Calculate cost
  - ➢ Backpropagation

Training refers to learning the parameters (*w* and *b*) of the model from the training data*

*Assumes ***X*** refers to training data with *m* rows and *d* columns

Training

Gradient Descent Algorithm

- ❖ Initialization
- ❖ Repeat until convergence:
  - ➢ Forward propagation
  - ➢ Calculate cost
  - ➢ Backpropagation

Initialize parameters ***w*** and *b*

- ➔ Create (d, 1) array ***w*** of random numbers
- ➔ Create variable *b* set to 0

Training

Loop for *max_iter* iterations

Gradient Descent Algorithm

- ❖ Initialization
- ❖ Repeat until convergence:
  - ➢ Forward propagation
  - ➢ Calculate cost
  - ➢ Backpropagation

Training

Gradient Descent Algorithm

- Initialization
- Repeat until convergence:
  - **Forward propagation**
  - Calculate cost
  - Backpropagation

Compute activations

→ $Z = X \cdot w + b$

→ $A = g(Z) = \text{sigmoid}(Z)$

Training

Gradient Descent Algorithm

- Initialization
- Repeat until convergence:
  - Forward propagation
  - Calculate cost
  - Backpropagation

Cost using current values of $\boldsymbol{w}$ and b

$$\frac{1}{m}\sum_{i=1}^{m} -\boldsymbol{y}\log(\boldsymbol{A}) - (1-\boldsymbol{y})\log(1-\boldsymbol{A})$$

Training

Gradient Descent Algorithm

- ❖ Initialization
- ❖ Repeat until convergence:
  - ➢ Forward propagation
  - ➢ Calculate cost
  - ➢ Backpropagation

Compute gradients

- ➔ $dZ = A - y$
- ➔ $dW = X^T \cdot dZ / m$
- ➔ $db = \frac{1}{m}\sum_{i=1}^{m} dZ$

Update parameters

- ➔ $W = W - \alpha \cdot dW$
- ➔ $b = b - \alpha \cdot db$

Testing

Testing refers to evaluating the trained model with testing data*

- Make predictions
- Assess how well predictions correspond to known labels

*Assumes ***X*** refers to testing data

Testing

- ❖ Make predictions
- ❖ Assess how well predictions correspond to known labels

<u>Predict activations</u>

- ➔ A = Forward propagation of X
- ➔ Predictions = Binarize (round) A

Testing

❖ Make predictions

❖ Assess how well predictions correspond to known labels

Score model

➔ Calculate percentage of predictions that correctly match labels

Overfitting

- ❖ *Overfitting* is one of the most common problems in ML
- ❖ Model learns properties specific to the training data that don't generalize to new (testing) data
- ❖ Performance is much better on training data than on testing data

Z





TS

Regularization

Smaller values for the parameters ***w*** lead to more generalizable models and are less prone to overfitting

To incentivize small values for ***w***, modify the cost function so that it:

1) Fits the training data well

and

2) Penalizes large values for ***w***

λ is regularization parameter

$$J = \frac{1}{m}\sum_{i=1}^{m} L + \frac{\lambda}{2m}\sum_{j=1}^{d} w_j^2$$

$$\frac{dJ}{dw} = \frac{1}{m}\sum_{i=1}^{m}(a-y)\cdot x + \frac{\lambda}{m}w$$